RECENT WORKS RELATED TO ALGEBRAIC VARIETIES AND AUTOMORPHISMS

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Abstract. Automorphisms of algebraic varieties begin to be studied from new view points based on log algebraic varieties. We are planning to hold a workshop in the year 2014 at RIMS which will focus on these new trends. In this article, we review recent works of the invitees to the workshop.

Recent trends of research related to the workshop

To make the situation very straightforward, the ground field k is assumed to be an algebraically closed field of characteristic zero. Whenever we make topological arguments depending on homology groups and fundamental groups, we understand that k is the complex number field $\mathbb{C}$.

In studying algebraic varieties of higher dimension, one effective approach is to decompose a given variety into algebraic varieties of lower dimension via a fiber structure. To find a fiber structure, the quotient morphism by an algebraic group action on the variety is expected to give a fibration whose fibers are the orbits of the algebraic group. Not only the quotient morphism but also the orbit stratification give a decomposition of the variety into orbit strata. As an example, we consider a similar decomposition by making use of an algebraic group action.

Let V be a smooth projective variety with a linear algebraic group G acting on it algebraically. If $\mathrm{Pic}\,(G) = (0)$, then any line bundle (identified with the associated divisor) is G-linearizable. If H is a very ample divisor, then G acts on the linear system $|H|$. If the G-action on $|H|$ has a fixed point corresponding to the very ample divisor H_0, the complement $X = V \setminus H_0$ is a smooth affine variety with the induced G-action and the structure of X can be detected by looking into a logarithmic pair (V, H_0). If H_0 is smooth, then H_0 has also the induced

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G-action. We can consider the induced ample divisor $H|_{H_0}$ on H_0 and repeat the above process to subvarieties of V with lower dimension.

Let G be a connected algebraic group which is not necessarily linear. Let V be a smooth projective variety with a G-action and let D be a G-stable effective divisor with simple normal crossings. The G-action induces a mapping of Lie algebras $\theta : \mathfrak{g} \to \Gamma(V, \mathcal{T}(-D))$ which is induced by an algebraic group homomorphism $G \to \mathrm{Aut}^0(V)$ [1].

Let G be a connected reductive group, B a Borel subgroup and U the unipotent radical of B. A normal algebraic variety X is called a *spherical variety* if X contains an open B-orbit. An example of a spherical variety is a projective homogeous space G/P, where P is a parabolic subgroup (i.e. a subgroup containing a Borel subgroup). In this case, G/P contains an open U-orbit. A toric variety is also spherical, i.e., $G = B = T$. Spherical varieties and their equivariant completions (or embeddings) are interesting problems (see [Brion 2012]).

As mentioned above, the homogeneous space G/P contains an open U-orbit with a unipotent group U. Expanding this case, we can define a *flexible* variety. Let X be a smooth algebraic variety. For every point $x \in X$, if the tangent space $\mathcal{T}_{X,x}$ is spanned by the vector fields of one-dimensional unipotent subgroups (i.e., the additive group G_a) of $\mathrm{Aut}(X)$, then X is called *flexible*. It is shown by Arzhantsev-Flenner-Kaliman-Kutzschbauch-Zaidenberg [AFKKZ 2013] that if X is a smooth affine flexible variety X is a homogeneous space of the subgroup $\mathrm{SAut}(X)$ of $\mathrm{Aut}(X)$ generated by one-dimensional unipotent subgroups [2].

These trends of research overlap with our direction of research to look into the structure of algebraic varieties via fibrations by the affine space (mostly the affine line $\mathbb{A}^1$ or the affine plane $\mathbb{A}^2$) or the actions by the additive group. If G_a acts on an affine variety Spec R, then $\Gamma(X, \mathcal{T}_X) \cong \mathrm{Der}_k(R)$ and the vector field associated to a G_a-action corresponds to a k-derivation of R called a *locally nilpotent derivation*. Similarly, a G_a-action on a smooth projective variety corresponds to a locally nilpotent vector field.

Note that if the quotient morphism $X \to X/G_a$ exists for a smooth affine variety X, i.e., the invariant subring $\Gamma(X, \mathcal{O}_X)^{G_a}$ is finitely generated, X contains an open $\mathbb{A}^1$-cylinder, whence $\overline{\kappa}(X) = -\infty$. The

[1] If θ is surjective (resp. bijective), we say that V is *log-homogeneous* (resp. *log-parallelizable*) along D. If V is log-homogenoeous, the complement $X = V \setminus D$ is a homogeneous space under G. By [Brion 2007, Theorem 3.2.1], the structure of complete (projective) log-homogeneous G-varieties can be described completely in terms of Levi subgroups of G.

[2] There is some similarity to a rationally connected variety.

following recent result of us [GKM 2013] is of some interest and indicates a direction of our research.

Theorem. *Let V be a smooth projective variety of dimension n and let H be a very ample divisor. Choose members $H_1, H_2, \ldots, H_n$ of $|H|$ such that $V_i = H_1 \cap H_2 \cap \cdots \cap H_n$ is a normal intersection and an irreducible subvariety of dimension $n-i$, where $0 \le i \le n$ and $V_0 = V$. Assume that the pair (V_i, V_{i+1}) has log kodaira dimension $-\infty$, i.e., $\overline{\kappa}(V_i \setminus V_{i+1}) = -\infty$ for $0 \le i \le n-1$. Then V is isomorphic to $\mathbb{P}^n$ and H is a hyperplane.*

When we consider the structure of the automorphism group $\mathrm{Aut}(\mathbb{A}^3)$, a one-dimensional unipotent group has some significance. The Nagata automorphism is given by

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2 z, y + (xz + y^2)z, z)$$

and realized as the element $\exp(\delta)$ of a one-dimensional unipotent subgroup $\{\exp(t\delta) \mid t \in k\} \cong G_a$, where δ is a locally nilpotent derivation defined by $\delta = (xz + y^2)\{-2y\partial_x + z\partial_y\}$. Contrary to the case of dimension two, the Nagata automorphism is a wild automorphism, i.e., not generated by linear and de Jonquière automorphisms (a theorem of Shestakov-Umirbaev, 2004) [3]. Hence it is expected that $\mathrm{SAut}(\mathbb{A}^3)$ should play some role in clarifying the structure of $\mathrm{Aut}(\mathbb{A}^3)$. Kuroda [Kuroda 2013] is engaged in this program.

Recent works related to the workshop

(1) Let V be a smooth algebraic variety. A general question is when V has a G_a-action. Kishimoto-Prokhorov-Zaidenberg [KPZ 2011] considered the case of the affine cone X_d of a del Pezzo surface S_d of degree d anti-canonically embedded in the projective space and show that the answer is negative if $d \le 2$ and positive if $4 \le d \le 9$. A subtle case is the case $d = 3$, and Flenner-Zaidenberg [FZ 2003] asked whether the affine hypersurface $x^3+y^3+z^3+w^3 = 0$ in $\mathbb{A}^4$ has an effective G_a-action. Recently, the case S_3 was settled negatively in [Cheltsov et al. 2013].

In [Dubouloz-Kishimoto 2012], they observed a smooth cubic hypersurface S in $\mathbb{P}^3$ with a hyperplane section $S \cap H$ consisting of a line and a conic meeting in one point with multiplicity two and showed that the affine threefold $X = \mathbb{P}^3 \setminus S$ has $\overline{\kappa}(X) = -\infty$ and a fibration $f : X \to \mathbb{A}^1$ such that a general fiber X_t has an $\mathbb{A}^1$-fibration but X has

[3] If one of the three variables is unchanged by an automorphism, then the automorphism is stably tame. Namely, increasing a number of variables, the extended automorphism becomes a tame automorphism [BEW 2012].

no $\mathbb{A}^1$-fibration. The proof is reduced finally to the irrationality of a smooth cubic hypersurface in $\mathbb{P}^4$ due to Clemens-Griffiths.

Inspired by this result, we considered in [GKM 2013 bis] deformations of $\mathbb{A}^1$-ruled affine surfaces and showed the difference between G_a-quotient morphisms and $\mathbb{A}^1$-fibrations on surfaces whose base curves are affine or projective. It is an honest impression of us that non-complete surfaces are still able to give interesting geometric problems when considered in the framework of higher-dimensional varieites.

An *additive structure* on a projective variety V of dimension n is an algebraic action $G_a^n \times V \to V$ such that V contains an open G_a^n-orbit. Since any homogeneous space of a unipotent group is isomorphic to the affine space, V contains a dense open set isomorphic to $\mathbb{A}^n$. Arzhantsev-Popovskiy [AP 2013] proved that if V is a complete variety such that $\mathrm{Aut}^0(V)$ is a reductive algebraic group and if V contains an open U-orbit, where U is a maximal unipotent subgroup of $\mathrm{Aut}^0(V)$, then V is a flag variety G/P, where G is a semi-simple algebraic group and P is a parabolic subgroup [4]. This area is fast-developing with the Russian young people working on it.

(2) Popov [Popov 2013] investigates a maximal torus in the Cremona group $\mathcal{C}_n$ which is the birational automorphism group of $\mathbb{P}^n$ and shows that if $n \geq 5$ there are no maximal tori of dimension $n-2, n-1$ and $> n$. The case of dimension n is not known. Blanc [Blanc 2011] classified the conjugacy classes of elements of $\mathcal{C}_2$ and cyclic finite groups of $\mathcal{C}_2$. He is one of the most active young mathematician in France and working mostly on various aspects of the Cremona groups. Moser-Jauslin with Dubouloz et al. [DMP 2011] proves a non-cancellation for a certain contractible affine threefold [5]. In [Moser-Jauslin 2011], she also worked out the structure of the automorphism group of the Koras-Russell contractible threefold which was introduced when they proved the linearizability of G_m on $\mathbb{A}^3$ in [Koras-Russell 1999]. This linearization of a G_m-action on $\mathbb{A}^3$ is a surprisingly profound result which uses a substantial body of the theory of open algebraic surfaces.

[4] In fact, they prove the result when U is commutative, i.e., $U \cong G_a^n$. But the proof applies to the non-commutative case, too.

[5] Even in dimension two, the cancellation problem has a counterexample. A typical one is a Danielewski surface $\{xy = z^2 - 1\}$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$, where Δ is the diagonal. This surface is also recognized as a homogeneous space $\mathrm{SL}(2)/T$, where T is a torus of $\mathrm{SL}(2)$.

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