RECENT DEVELOPMENTS IN AFFINE ALGEBRAIC GEOMETRY

FROM THE PERSONAL VIEWPOINTS OF THE AUTHOR

MASAYOSHI MIYANISHI

Abstract. We shall review recent developments in affine algebraic geometry. The topics treated in the present article cover only a part of this vast area of research.

Contents

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0. Introduction

There is no clear definition of affine algebraic geometry. It is one branch of algebraic geometry which deals mostly with the affine spaces and the polynomial rings, hence affine algebraic varieties as subvarieties of the affine spaces and finitely generated algebras as the residue rings of the polynomial rings.

In affine algebraic geometry, affine varieties and affine domains, i.e., finitely generated algebras which are integral domains, are closely related in methods and applications. Geometric observations are to be applied to affine domains to obtain results which are stated in terms of commutative algebras and vice versa.

In the present article, the author intends to review the recent developments in affine algebraic geometry. But the materials to be treated must have influences from the personal viewpoints and interests. We note that the important problems that affine algebraic geometry was in face with as of the 1990's are summarized in a survey article of Kraft [163].

The author is almost ignorant of the earlier stages of affine algebraic geometry. To him, the first important result seems to be a theorem attributed to Jung [120] (and called the *automorphism theorem*) about the structure of the automorphism group of the affine plane, by which the group is an amalgamated product of the group of affine transformations and the group of de Jonquière transformations. As explained in Nagata [222], the proofs published before van der Kulk [297] including Jung's original one are rather difficult to follow. But the proof of van der Kulk is based on the observation that if an automorphism is not linear then the transforms of the lines under the automorphism are the curves having one place in a common point at infinity and he used this observation to reduce the degree of the transformed curves.

In 1975, Abhyankar-Moh [6] published a proof of the embedding line theorem. It states that if an embedding $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ is given by polynomials $x = f(t)$ and $y = g(t)$ of respective degrees m and n and if either m or n is not divisible by the characteristic of the field then either m divides n or n divides m. Around the same period, Suzuki [281] proved independently the same result in analytic methods. In fact, the embedding line theorem yields a proof of the automorphism theorem. Study of curves on $\mathbb{A}^2$ isomorphic to $\mathbb{A}^1$ was thus an object of great concerns and one of the driving forces in affine algebraic geometry. The embedding line theorem (now called *AMS theorem* after Abyhankar-Moh and Suzuki) has now several different proofs (cf. [185, 186, 251, 248, 145, 13, 91, 8, 88, 318]). The core of the proofs (algebraic, geometric or topological) is an analysis of the one-place singularity of the embedded line at infinity. This kind of approach leads to a study of a pencil of curves on the affine plane or affine algebraic surfaces such that the possible base points lie on the boundary at infinity and the general members have one-places at infinity. The elimination of base points lying on the boundary at infinity via a sequence of blowing-ups will produce a pencil of curves on the affine algebraic surfaces whose general members are isomorphic to the affine line, which is called an $\mathbb{A}^1$-*fibration*. The notion of $\mathbb{A}^1$-fibration has significance in later development, though it looks very simple. More generally, if C_0 is an irreducible plane curve with only one place at infinity and with the defining equation $f = 0$, let $\Lambda(f)$ be the pencil consisting of the curves C_α defined by $f = \alpha$ for all $\alpha \in k$, where k is an algebraically closed ground field. In the case of $\mathrm{char}\,(k) = 0$, every member C_α has only one place at infinity [5, 213] (*Irreducibility theorem*). If $C_0 \cong \mathbb{A}^1$ then $C_\alpha \cong \mathbb{A}^1$ for all $\alpha \in k$. In the case of $\mathrm{char}\,(k) > 0$, general members of $\Lambda(f)$ have one places at infinity [79] (*Generic irreducibility theorem*). In the latter case, if C_0 is isomorphic to $\mathbb{A}^1$ one can then ask if the generic member is a form of $\mathbb{A}^1$ defined over the rational function field $k(t)$. Study of forms of $\mathbb{A}^1$ or the additive group scheme G_a leads to unirational surfaces.

The solution of the Serre conjecture by Quillen [245] and Suslin [280] in 1976 had vast influences as a basic result on the developments of various branches in affine algebraic geometry.

As a generalization of the AMS theorem, there is a challenging working conjecture (called the *Abhyankar-Sathaye conjecture*) which asserts that any closed subvariety of $\mathbb{A}^n$ isomorphic to $\mathbb{A}^r$ for $1 \leq r < n$ is a linear subspace after the conjugation by a certain automorphism of $\mathbb{A}^n$.

Another guiding problem in the 1970's was the *cancellation problem.* It was first mentioned in the form : If A and B are k-algebras such that $A[t] \cong B[u]$ with variables t, u, does it hold that $A \cong B$? The birational case asking whether $K(t) \cong L(u)$ implies $K \cong L$ is called *Segre Problem* or *Zariski problem.* Hence the biregular case is also called the *Zariski Problem.* Most interesting is the case where B is a polynomial ring $k[x_1, \ldots, x_n]$ over an algebraically closed field k. The case $n = 1$ is rather easy. In fact, A is then a unique factorization domain with $A^* = k^*$, whence $A \cong k[x_1]$. The problem becomes tough when $n \geq 2$. The case $n = 2$ has an affirmative answer due to Fujita [76] and the author-Sugie [202]. The proof depends on an algebraic characterization of $\mathbb{A}^2$ as an algebraic variety and a result for the affine-ruledness of algebraic surfaces which corresponds to Enriques' criterion of ruledness for projective surfaces. The latter result depends on the theory of *logarithmic Kodaira dimension*, which was introduced by Iitaka [114] and other people and became a guiding principle in construction of the so-called theory of open algebraic surfaces modelled on the theory of projective surfaces (see Kawamata [146] and the author [188, 195]). The cancellation problem itself has counterexamples, the first one of which was given by Danielewski [42] (see Fieseler [61]). Danielewski's example was generalized by tom Dieck [290], Wilkens [301] and Masuda-the author [175]. The cancellation problem for complete varieties was also treated (see [115] for a positive result in the case of non-negative Kodaira dimension).

With the theory of open algebraic surfaces being prepared, applications of the theory to various problems got started. One of the applications is to homology planes which was originated by a striking result of Ramanujam [246] but was dormant for nearly twenty years when the paper [94] was published in 1988. Thence the rationality of homology planes was proved by Gurjar-Shastri [103, 104] and the structures of these surfaces were revealed by people like Gurjar, Petrie, Pradeep, Shastri, Sugie, tom Dieck and the author. Another application seems to be in the theory of transformation groups when the quotient varieties are algebraic surfaces. A significant contribution of Koras-Russell [161, 160] to the linearizability of a torus action on the affine 3-space is benefitted from the theory of open algebraic surfaces. There are more applications, but the author would like to mention only the possibility of applying the theory to the (generalized) Jacobian conjecture. The author considers it very significant to approach the conjecture from a more geometric point of view. Such an approach has a possibility of avoiding the complexity to handle polynomials in many variables.

In what follows we fix a ground field k which is assumed to be algebraically closed unless otherwise specified. The characteristic of k is zero in most cases, but we do make some observations in the case of positive characteristic p as well.

1. Various aspects of the additive group scheme actions and related results

1.1. **Locally nilpotent derivations.** Let G_a be the additive group scheme and G_m the multiplicative group scheme. Over the ground field k, G_a (resp. G_m) has the coordinate ring $k[t]$ (resp. $k[t,t^{-1}]$) which is a polynomial ring in one variable (resp. a Laurent polynomial ring in one vaiable). The group law on G_a is given by a triplet $(\Delta, \varepsilon, \iota)$, which consists of k-algebra homomorphisms called respectively the comultiplication, the augmentation and the coinverse and defined as follows.

$$\begin{aligned} \Delta : k[t] \to k[t] \otimes_k k[t], &\qquad \Delta(t) = t \otimes 1 + 1 \otimes t \\ \varepsilon : k[t] \to k, &\qquad \varepsilon(t) = 0 \\ \iota : k[t] \to k[t], &\qquad \iota(t) = -t \end{aligned}$$

The group law for G_m is given by

$$\begin{aligned} \Delta : k[t,t^{-1}] \to k[t,t^{-1}] \otimes_k k[t,t^{-1}], &\qquad \Delta(t) = t \otimes t \\ \varepsilon : k[t,t^{-1}] \to k, &\qquad \varepsilon(t) = 1 \\ \iota : k[t,t^{-1}] \to k[t,t^{-1}], &\qquad \iota(t) = t^{-1} \end{aligned}$$

Let $G = \operatorname{Spec} R$ be an affine group k-scheme defined by $(\Delta, \varepsilon, \iota)$, where $\Delta : R \to R \otimes_k R$, $\varepsilon : R \to k$, $\iota : R \to R$ are k-algebra homomorphisms. If G acts on an affine scheme $X = \operatorname{Spec} A$ by $\sigma : G \times X \to X$, σ corresponds to the coaction $\varphi : A \to R \otimes_k A$ such that (1) $(\Delta \otimes \mathrm{id}_A) \cdot \varphi = (\mathrm{id}_R \otimes \varphi) \cdot \varphi$ and (2) $(\varepsilon \otimes \mathrm{id}_A) \cdot \varphi = \mathrm{id}_A$.

In the case $G = G_a$, we identify $k[t] \otimes A$ with a polynomial ring $A[t]$ and write $\varphi : A \to A[t]$ as

$$\varphi(a) = \sum_{i=0}^{\infty} \delta_i(a) t^i \quad \text{for } a \in A, \tag{1.1.1}$$

where δ_i is a k-module homomorphism of A. Then the following result is well-known (cf. [186, Chap.I, §1]).

Lemma 1.1.1. *With the notations as above, the following assertions hold.*

(1) *If* $\operatorname{char}(k) = 0$ *then* δ_1 *is a locally nilpotent* k*-derivation on* A *and* $\delta_i = (\delta_1)^i / i!$ *for* $i \geq 0$*, where* $\delta_0 = \mathrm{id}_A$. Here a k-derivation D on A is *locally nilpotent* if given $a \in A$, $D^n(a) = 0$ for $n \gg 0$.

(2) *If* char $(k) = p > 0$ *then* $D = \{\delta_0, \delta_1, \dots\}$ *is a locally finite iterative higher derivation on* A (see [186, Chap.I,§1] for the definition) *and*

$$\delta_i = \frac{(\delta_1)^{i_0}(\delta_p)^{i_1}\cdots(\delta_{p^r})^{i_r}}{(i_0)!(i_1)!\cdots(i_r)!},$$

where $i = i_0 + i_1 p + \cdots + i_r p^r$ *is the* p*-adic expansion of* i.

Hence, in the case of char $(k) = 0$, the coaction $\varphi : A \to R \otimes A$ in (1.1.1) is written as

$$\varphi(a) = \sum_{i=0} \frac{1}{i!} D^i(a) t^i \quad \text{for } a \in A. \tag{1.1.2}$$

where $D = \delta_1$. In what follows, whenever we speak of a locally finite iterative higher derivation $D = \{\delta_0, \delta_1, \dots\}$, we identify, in the case char $(k) = 0$, D with a locally nilpotent derivation represented by δ_1. By making use of (1.1.1), we can extend the G_a-action on A to the quotient field $K := Q(A)$ by setting

$$\Psi(\frac{b}{a}) = \frac{\varphi(b)}{\varphi(a)} \in K[[t]] \quad \text{for } a, b \in A \text{ with } a \neq 0. \tag{1.1.3}$$

For $\lambda \in k$, define ${}^\lambda a = \varphi(a) \mid_{t=\lambda}$. Then ${}^\lambda(b/a) = {}^\lambda b/{}^\lambda a$.

In the case $G = G_m$, we identify $k[t, t^{-1}] \otimes A = A[t, t^{-1}]$ and write $\varphi : A \to A[t, t^{-1}]$ as

$$\varphi(a) = \sum_{i=-\infty}^{\infty} \chi_i(a) t^i \quad \text{for } a \in A. \tag{1.1.4}$$

Then it is easy to show that $\chi_i \chi_j = \delta_{ij} \chi_i$ for $i, j \in \mathbb{Z}$ and $\sum_i \chi_i = \mathrm{id}_A$. Furthermore, $\chi_\ell(ab) = \sum_{i+j=\ell} \chi_i(a)\chi_j(b)$ for $a, b \in A$. Let $A_i = \chi_i(A)$. Then these relations imply that $A = \bigoplus_{i=-\infty}^{\infty} A_i$ is a direct sum decomposition and $A_i \cdot A_j \subset A_{i+j}$. Hence A is a $\mathbb{Z}$-graded k-algebra. Conversely, if given such a $\mathbb{Z}$-graded k-algebra $A = \bigoplus_{i=-\infty}^{\infty} A_i$, define $\chi_i : A \to A$ by a composite of the projection $A \to A_i$ and the inclusion $A_i \hookrightarrow A$. Then we can readily show the above relations among the χ_i's. Hence we obtain the folowing result.

Lemma 1.1.2. *A* G_m*-action on an affine scheme* $X = \operatorname{Spec} A$ *gives rise to a* $\mathbb{Z}$*-graded* k*-algebra structure on* A *and vice versa.*

The ring of invariants under the G-action is given by $A_0 = \operatorname{Ker} \delta_1$ if char $(k) = 0$ and $G = G_a$, $A_0 = \{a \in A \mid \delta_{p^i}(a) = 0, \ \forall i \geq 0\}$ if char $(k) = p > 0$ and $G = G_a$ and $A_0 = \chi_0(A)$ if $G = G_m$. In the case of G_m-action as well as a reductive algebraic group action, it is known by a theorem of Hilbert that A_0 is finitely generated provided so is A.

It is noteworthy that the converse holds. Namely, a result of Popov [241] asserts that, given an algebraic group G, if the ring of invariants of any algebraic G-action on $\operatorname{Spec} A$ with variable A finitely generated is finitely generated, then G is reductive. In the case of G_a-action, we set $K_0 = \{z \in K \mid \Psi(z) = z\}$. Then K_0 is a subfield, $A_0 = K_0 \cap A$ and K_0 is the quotient field of A_0 (see [186, Lemma 1.3.2]). We have the following result of Zariski (cf. [221, p.52]).

Lemma 1.1.3. *Let A be an affine domain with a non-trivial G_a-action. Then A_0 is finitely generated if either* $\mathrm{tr.deg}_k K_0 = 1$ *or A is a normal domain and* $\mathrm{tr.deg}_k K_0 = 2$.

If $\dim A \geq 5$ then $\dim A_0 \geq 4$ and A_0 is not necessarily finitely generated over k as shown by the counterexamples to the fourteenth problem of Hilbert (cf. 1.3 below).

Let $q : X \to B$ be a morphism induced by the natural inclusion $A_0 \hookrightarrow A$, which is called the algebraic quotient morphism under the G-action on X. We denote B by $X/\!/G$. In the case of G_a-actions, the quotient morphism can be elucidated via the following result [200].

Lemma 1.1.4. *Let $D = \{\delta_0, \delta_1, \ldots\}$ be a non-trivial (i.e., $A \neq A_0$), locally finite, iterative higher derivation on an integral domain over k. Then the following assertions hold.*

(1) *If there is an element ξ of A such that $\delta_1(\xi) = 1$ and $\delta_i(\xi) = 0$ for all $i > 1$, then $A = A_0[\xi]$ and ξ is algebraically independent over A_0.* (This element ξ is called a *slice*.)
(2) *In general, there exist elements $c \neq 0$ of A_0 and ξ of A such that $A[c^{-1}] = A_0[c^{-1}][\xi]$.* (This element ξ is called a *local slice*.)

This lemma implies that the quotient morphism $q : X \to X/\!/G_a$ defines an $\mathbb{A}^1$-fibration on X (see §2 for $\mathbb{A}^1$-fibrations). The properties of the ring of invariants under a G_a-action is summarized as follows.

Lemma 1.1.5. *Let A be an integral domain over k and let $D = \{\delta_0, \delta_1, \ldots\}$ be a non-trivial locally finite iterative higher derivation on A. Let A_0 be the ring of invariants. Then we have:*

(1) *A_0 is an inert subring of A. Namely, if $ab \in A_0$ for $a, b \in A$ then $a, b \in A_0$. Hence A_0 is a unique factorization domain provided so is A.*
(2) *$A^* = A_0^*$, where A (resp. A_0^*) denotes the multiplicative group of invertible elements of A (resp. A_0^*).*

A locally finite iterative higher derivation is closely connected with a polynomial ring in one variable as shown in the following result [180].

Lemma 1.1.6. *Let K be a field of arbitrary characteristic and let $D = \{\delta_0, \delta_1, \dots\}$ be an iterative higher derivation on K. Namely a mapping defined by*

$$\varphi : K \to K[[t]], \quad \varphi(a) = \sum_{i=0}^{\infty} \delta_i(a)t^i, \quad a \in K$$

is a ring homomorphism satisfying $\delta_i\delta_j = {}_{i+j}C_i\delta_{i+j}$ for all $i, j \geq 0$ and $\varepsilon \cdot \varphi = \mathrm{id}_K$, where a K-homomorphism $\varepsilon : K[[t]] \to K$ is given by $\varepsilon(t) = 0$ and ${}_{i+j}C_i$ is a binomial coefficient. Let $R = \varepsilon(\varphi(K) \cap K[t])$, which consists of elements a of K such that $\varphi(a)$ is a polynomial in t. Let $k = \{a \in K; \delta_i(a) = 0, \forall i > 0\}$. Then k is a subfield of K and either R is a polynomial ring in one variable over k or reduced to k.

1.2. **Actions of G_a and G_m on the affine spaces.** In this subsection, we assume the ground field k to be algebraically closed. First of all, let us consider a G_a-action on the affine space $\mathbb{A}^n$. It can be explicitly described if the dimension n is small. The first result in this direction is a result of Rentschler [247].

Theorem 1.2.1. *Suppose that* $\mathrm{char}\,(k) = 0$. *For a G_a-action on the affine plane $\mathbb{A}^2$, we find a system of coordinates $\{x, y\}$ on $\mathbb{A}^2$ such that $\varphi(x) = x$ and $\varphi(y) = y + f(x)t$ with $f(x) \in k[x]$, where $\varphi : k[x, y] \to k[x, y][t]$ is the coaction.*

The proof of this result is now very easy. In fact, the ring of invariants A_0 is a UFD of dimension one and $A_0^* = k^*$. Hence A_0 is a polynomial ring in one variable $k[x]$. The quotient morphism $q : \mathbb{A}^2 \to \mathbb{A}^1 = \mathrm{Spec}\, A_0$ defines an $\mathbb{A}^1$-fibration. By the algebraic characterization of the affine plane (see Theorem 2.2.3), we can choose the element ξ in Lemma 1.1.4, (2) so that the coordinate ring A of $\mathbb{A}^2$ is isomorphic to $k[x, \xi]$. Let $y = \xi$. Then the coaction φ of the given G_a-action is given as stated.

The result of Rentschler can be modified to fit to the case of positive characteristic [181].

Theorem 1.2.2. *Suppose that* $\mathrm{char}\,(k) = p > 0$. *For a G_a-action on the affine plane $\mathbb{A}^2$, we can find a system of coordinates $\{x, y\}$ such that the coaction φ is given by*

$$\begin{aligned} \varphi(x) &= x \\ \varphi(y) &= y + f_0(x)t + f_1(x)t^p + \cdots + f_r(x)t^{p^r}, \end{aligned}$$

where $f_0(x), f_1(x), \dots, f_r(x) \in k[x]$.

For the case $n \geq 3$, not many results, though important, are available. We mention some of the interesting results.

Theorem 1.2.3. *Suppose that* char $(k) = 0$. *Let* σ *be a non-trivial* G_a-*action on the affine* 3-*space* $\mathbb{A}^3_{\mathbb{C}}$. *Then the following assertions hold.*

(1) *The ring of invariants is isomorphic to a polynomial ring in two variables* (the author [190]).
(2) *Suppose that* $k = \mathbb{C}$. *The quotient morphism* $q : \mathbb{A}^3 \to \mathbb{A}^2$ *is then surjective* (Bonnet [37]). *Furthermore, q is smooth* (Deveney-Finston [47]) [1].
(3) *Suppose that* $k = \mathbb{C}$ *and that the action* σ *is free, i.e.,* σ *has no fixed points. Then* $q : \mathbb{A}^3 \to \mathbb{A}^2$ *is a trivial* $\mathbb{A}^1$-*bundle. Hence we can choose a system of coordinates* $\{x, y, z\}$ *in such a way that* $\varphi(x) = x$, $\varphi(y) = y$ *and* $\varphi(z) = z + f(x,y)t$, *where* φ *is the coaction of* σ *and* $f(x,y) \in k[x,y,z]$ (Kaliman [125]) [2].

These are fairly deep results which depend on the understanding of the affine plane as an algebraic variety, topology of polynomial mappings and degenerations of the affine line. These backgrounds will be discussed in later sections. The assertion (1) does not hold in the case of higher dimension (see Winkelmann [302] for a free triangular G_a-action on $\mathbb{A}^5$ whose quotient is not isomorphic to $\mathbb{A}^4$). The assertion (2) does not hold either (see Bonnet[37] for a concrete G_a-action on $\mathbb{A}^4$ whose quotient is $\mathbb{A}^3$ but quotient morphism is not surjective). There is a survey article by Snow [273] concerning the unipotent group actions on the affine spaces. A very comprehensive account of locally nilpotent derivations and G_a-actions is to be published by Freudenburg[73]. There is a well-organized explanations of the subject with its history and a wide range of references.

Comments. We assume that char $(k) = 0$ for the sake of simplicity. Let A be an affine k-domain which we assume to be factorial, that is, a unique factorization domain. Suppose that G_a acts non-trivially on $X = \operatorname{Spec} A$. Let D be the corresponding locally nilpotent derivation of A and let $A_0 = \operatorname{Ker} D$. Though A_0 is not necessarily finitely generated as shown by the counterexamples of the fourteenth problem of Hilbert, it is expected that A_0 enjoys, to some extents, the same properties as A. We shall list below some of the properties that A_0 has and raise some questions.

(1) A_0 *is a unique factorization domain by Lemma 1.1.4. Hence every height one prime ideal of* A_0 *is principal, and every prime divisor, which is finite in number, of a nonzero element has height one.*

[1] See Kraft [164] for a generalization of this result

[2] See Kaliman-Saveliev [128] and Kraft [164] for a generalization of this result to the case where $\mathbb{A}^3$ is replaced by a smooth contractible threefold with a free G_a-action

(2) *The quotient morphism* $q : X \to X//G_a$ *is surjective in codimension one. That is to say, for any prime ideal* $\mathfrak{p}$ *of* A_0 *of height one, there exists a height one prime ideal* $\mathfrak{P}$ *of* A *such that* $\mathfrak{P} \cap A_0 = \mathfrak{p}$.
(3) *Does the chain condition for prime ideals of* A_0 *hold ?*
(4) *Are all projective modules over* A_0 *free provided* A *is a polynomial ring over* k *?*

Instead of locally nilpotent derivations, one can consider a k-derivation δ on an affine k-domain A. It is possible to regard δ as a regular vector field on $X := \operatorname{Spec} A$. We can thus consider the invariant subring of δ. We just mention several references in this direction, which are, for example, Bass [27], the author [198] and Aoki-the author [11].

We shall next consider an algebraic torus action on the affine space. The problem was considered in the following context.

Linearization Problem. *Let* G *be a reductive algebraic group and let* $\sigma : G \times \mathbb{A}^n \to \mathbb{A}^n$ *be an algebraic action of* G *on the affine* n*-space. Does there then exist an automorphism* α *of* $\mathbb{A}^n$ *such that* $\alpha \cdot \sigma \cdot (\mathrm{id}_G \times \alpha)^{-1} : G \times \mathbb{A}^n \to \mathbb{A}^n$ *is a linear representation ?*

This problem was raised by Kambayashi [135] as a generalization of the fact that an action of a linearly reductive algebraic group on $\mathbb{A}^2$ is conjugate to a linear action [135]; see also [267] and [186, Theorem 3.5] for the case of a finite group action. We do not go into details of this problem in the general context. We just note that there are several results in the affirmative and that the first counterexample was given by Schwarz [266] with the complex orthogonal group $G = \mathrm{O}_2 = G_m \rtimes \mathbb{Z}/2$ acting on $\mathbb{A}^4$.

If we confine ourselves to the case $G = G_m$, the following results were obtained.

(1) *Any* G_m*-action on* $\mathbb{A}^2$ *is linearizable* (Gutwirth [108]).
(2) *Any torus action on* $\mathbb{A}^n$ *has a fixed point* (Białynicki-Birula [33]).
(3) *Any effective action of* G_m^{n-1} *on* $\mathbb{A}^n$ *is linearizable* (Białynicki-Birula [34]).
(4) *If an action of* $T = G_m^r$ *on* $\mathbb{A}^n$ *is unmixed and the fixed point locus* $(\mathbb{A}^n)^T$ *has dimension* ≤ 2*, then the action is linearizable* (Kambayashi-Russell [142]).
(5) *Any* G_m*-action on* $\mathbb{A}^3$ *is linearizable* (Koras-Russell [161]).
The proof of this result is deep and has possibilities of pushing forward researches on contractible threefolds and the Makar-Limanov invariant. Crucial is the case of hyperbolic G_m-action.

Then the smooth part of the quotient variety $\mathbb{A}^3//G_m$ has logarithmic Kodaira dimension $-\infty$. To prove this result, the authors make essential uses of the theory of open algebraic surfaces. Furthermore, Popov [243] used this result to show that any connected[3] reductive algebraic group action on $\mathbb{A}^3$ is linearizable.[4]

(6) *If k is non-closed, a torus action on $\mathbb{A}^n$ is not necessarily linearizable.* In fact, Asanuma [17] proved that *over any field k, if there exists a non-rectifiable closed embedding of $\mathbb{A}^m_k$ into $\mathbb{A}^n_k$, then there exist non-linearizable faithful actions of $(k^*)^r$ on $\mathbb{A}^{1+n+m}$ for $1 \leq r \leq 1+m$. Over the real field $\mathbb{R}$, there is a non-rectifiable embedding of $\mathbb{R}^1$ into $\mathbb{R}^3$* [269].[5]

(7) *Let k be an infinite field of positive characteristic. Then there are examples of non-linearizable torus actions on $\mathbb{A}^n$* (Asanuma [18]).

In the case $G = G_a$, any G_a-action on $\mathbb{A}^2$ is linearizable by Rentschler [247]. By Bass [24], G_a-actions on $\mathbb{A}^3$ are not necessarily linearizable, nor triangularizable. This result of Bass is extended to $\mathbb{A}^n$ for any $n \geq 3$ in Popov [242].

1.3. **The fourteenth problem of Hilbert.** Throughout this subsection, the ground field k is assumed to be an algebraically closed field of characteristic zero. We shall treat this topic here because the counterexamples are mostly obtained as the ring of invariants under certain unipotent group actions. Hence we discuss the portion of the topic which is concerned with the unipotent group actions. One can find a very nice explanation of the problem in Nagata [221]. There are excellent survey articles by Mumford [218] and Freudenberg [72, 73].

The fourteenth problem of Hilbert. *Let $k[x_1, \ldots, x_n]$ be a polynomial ring and let L be a subfield of $k(x_1, \ldots, x_n)$. Is the ring $L \cap k[x_1, \ldots, x_n]$ then finitely generated ?*

Let A be a normal affine domain over k and let L be the quotient field of A. For an ideal $\mathfrak{a}$, define the $\mathfrak{a}$-*transform* by $S(\mathfrak{a}; A) = \{z \in L \mid z\mathfrak{a}^n \subseteq A \text{ for some } n > 0\}$. Then $S(\mathfrak{a}; A) = \bigcap_{\mathfrak{p} \in J} A_{\mathfrak{p}}$, where J is the set of height 1 prime ideals of A such that $\mathfrak{p} \not\supset \mathfrak{a}$. So, $S(\mathfrak{a}; A) =$

[3]Popov pointed out the importance of this hypothesis to the author. For example, the linearizability of a finite group action on $\mathbb{A}^3$ is not known.

[4]There is a comprehensive overview of this result in [126] and there is also a featured review by the author [196] which overviews the historical backgrounds and the related results.

[5]A closed embedding $\iota : \mathbb{A}^k \to \mathbb{A}^n$ is said to be *rectifiable* if it is conjugate to a linear embedding by an automorphism of $\mathbb{A}^n$.

$\Gamma(X - V, \mathcal{O}_X)$, where $X = \operatorname{Spec} A$ and $V = V(\mathfrak{a})$. The following result gives a geometric interpretation of the problem.

Lemma 1.3.1. [221, p. 45] *Let R be a normal k-affine domain with the quotient field K and let L be an intermediate field of K/k. Let $R' = R \cap L$. Then there exist a normal affine domain A and an ideal $\mathfrak{a}$ of A such that $R' = S(\mathfrak{a}; A)$.*

There are a number of contributions to the problems. In most cases, the subfield L in the problem is obtained as the invariant subfield of $k(x_1, \ldots, x_n)$ when an algebraic group G acts on $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$ algebraically. Hence the finite generation of the ring of invariants $k[x_1, \ldots, x_n]^G$ is forcussed on. Let U be the unipotent radical of G. By a theorem of Hilbert, $k[x_1, \ldots, x_n]^G$ is finitely generated if so is $k[x_1, \ldots, x_n]^U$. Hence the actions of unipotent groups will be keys to determine the finite generation. There is one result in the affirmative.

(1) *If G_a acts linearly on the complex affine n-space $\mathbb{A}^n$, then the ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^{G_a}$ is finitely generated* (Weitzenböck [268]).

There are now many counterexamples, among which the first ones (see the result (2) below) were discovered by Nagata [219].

(2) *Let G_a^n act on $S := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $x_i \mapsto x_i, y_i \mapsto y_i + t_i x_i$ for $1 \le i \le n$. Let $n = 16$ and let G be a general linear subgroup of G_a^n of codimension* 3*. Then S^G is not finitely generated* [219]. [6]
(3) *Let G_a act non-linearly on $\mathbb{A}^7 = \operatorname{Spec} k[X, Y, Z, S, T, U, V]$ via a locally nilpotent derivation*

$$\Delta = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V},$$

where $t \ge 2$. Then the ring of invariants is not finitely generated (Roberts [249]). [7]
(4) *Let G_a act non-linearly on $\mathbb{A}^6 = \operatorname{Spec} k[X, Y, S, T, U, V]$ via a locally nilpotent derivation*

$$\Delta = X^3\frac{\partial}{\partial S} + Y^3 S\frac{\partial}{\partial T} + Y^3 T\frac{\partial}{\partial U} + X^2Y^2\frac{\partial}{\partial V}.$$

---

[6] A similar counterexample was later obtained by Steinberg [275] when $n = 9$ and $\dim G = 6$. There is a linear counterexample due to A'Campo-Neuen [7] in which G_a^{12} is acting on $\mathbb{A}^{19}$. There is a further development in this direction by Mukai [217].

[7] In the original proof of Roberts, the role of G_a-action was not clear. There is a clarification of this role as well as a generalization of the counterexample in the case of a higher number of variables [159].

Then the ring of invariants is not finitely generated (Freudenburg [71]).

(5) *Let G_a act non-linearly on $\mathbb{A}^5 = \operatorname{Spec} k[a, b, x, y, z]$ via a locally nilpotent derivation*

$$\Delta = a^2 \frac{\partial}{\partial x} + (ax + b)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

Then the ring of invariants is not finitely generated (Daigle-Freudenburg [43]). [8]

(6) *If $n = 3$ or 4, there exists an intermediate field L/k of $k(X_1, \ldots, X_n)$ such that $L \cap k[X_1, \ldots, X_n]$ is not finitely generated* (Kuroda [165, 166]). These examples of Kuroda give the counterexamples of the smallest possible dimension which is 3.

1.4. **Infinitesimal group schemes and Galois theory of purely inseparable extensions.** We assume $\operatorname{char}(k) = p > 0$ throughout this subsection. We can define infinitesimal group schemes α_p and μ_p as those with the common coordinate ring $\operatorname{Spec} k[x]/(x^p)$ but with $(\Delta, \varepsilon, \iota)$ defined by $\Delta(x) = x\otimes 1+1\otimes x$ for α_p, $\Delta(x) = x\otimes 1+1\otimes x+x\otimes x$ for μ_p, $\varepsilon(x) = 0$ and $\iota(x) = -x$, where we identify x with the residue class of x modulo (x^p). In fact, α_p (resp. μ_p) is the kernel of the Frobenius endomorphism of G_a (resp. G_m).

In order to describe the actions of α_p and μ_p, we shall recall p-Lie algebras. Let $\mathcal{L}$ be a Lie algebra defined over k with bracket product. If $\mathcal{L}$ has further a p-th power operation $D \mapsto D^p$ and satisfies $(cD)^p = c^p D^p$ for $c \in k$, we call $\mathcal{L}$ a p-Lie algebra. For a k-algebra A, the k-module $\operatorname{Der}(A)$ of all k-derivations of A into itself is a p-Lie algebra with bracket product $[D_1, D_2](a) = D_1(D_2(a)) - D_2(D_1(a))$ and p-th power operation $D^p(a)$, where $D^i(a) = D(D^{i-1}(a))$ for $1 \le i \le p$. Note that $\operatorname{Der}(A)$ is an A-module by $(aD)(b) = aD(b)$ for $a, b \in A$.

If α_p (resp. μ_p) acts on an affine k-scheme $X = \operatorname{Spec} A$ then the coaction $\varphi : A \to k[x] \otimes A$ is given by

$$\varphi(a) = \sum_{i=0}^{p-1} \delta_i(a) x^i \quad \text{for } a \in A, \tag{1.4.1}$$

where $x^p = 0$. Then $\delta_0 = \mathrm{id}_A$ and $D := \delta_1$ is a k-derivation of A into itself. Hence $D \in \operatorname{Der}(A)$ and $D^p = 0$ (resp. $D^p = D$) in the case of α_p-action (resp. μ_p-action). An element D of a p-Lie algebra $\mathcal{L}$ is said to be of *additive* (resp. *multiplicative*) type if $D^p = 0$ (resp.

[8] Tanimoto [285] has shown that a polynomial ring in one variable over the ring of invariants due to Daigle-Frudenburg is obtained as the ring of invariants of $k[X_1, \ldots, X_{13}]$ under a linear action of a non-commutative unipotent group $G_a^8 \rtimes G_a$.

$D^p = D$). Thus, a nonzero k-derivation $D \in \mathrm{Der}\,(A)$ of additive (resp. multiplicative) type corresponds to a non-trivial α_p (resp. μ_p)-action on $X = \mathrm{Spec}\,A$. Historically, p-Lie algebras were first considered to describe the Galois correspondence of purely inseparable extensions of height one. Namely we have the following well-known result [117].

Lemma 1.4.1. *Let K/k be a finitely generated field extension and let K^p be the subfield generated by all elements a^p over k when a ranges over elements of K. Then there is a one-to-one correspondence between the subfields L/K^p of K/K^p and the p-Lie subalgebras $\mathcal{L}$ of $\mathrm{Der}\,(K)$, which is given by $L \mapsto \{D \in \mathrm{Der}\,(K); D|_L = 0\}$ and $\mathcal{L} \mapsto \{\xi \in K; D(\xi) = 0$ for all $D \in \mathcal{L}\}$.*

The notion of higher derivation was introduced to extend this result to purely inseparable extensions of height larger than one (cf. [110] for example).

We shall discuss forms of $\mathbb{A}^1$ and G_a. An algebraic variety X defined over a field k is a *form of the affine n-space* $\mathbb{A}^n$ if $X \otimes_k \overline{k} \cong \mathbb{A}^n_{\overline{k}}$, where $\overline{k}$ is an algebraic closure of k. Then $X \otimes_k \overline{k} \cong \mathbb{A}^n_{\overline{k}}$ if and only if $X \otimes_k k' \cong \mathbb{A}^n_{k'}$ for a finite extension k'/k, where $X \otimes_k \overline{k}$ or $X \otimes_k k'$ signifies $\mathrm{Spec}\,(A \otimes_k \overline{k})$ or $\mathrm{Spec}\,(A \otimes_k k')$ when $X = \mathrm{Spec}\,A$. If k'/k is a separable Galois extension of group G, the separable forms of $\mathbb{A}^n$ are classified by the Galois cohomology $H^1(G, \mathrm{Aut}\,(\mathbb{A}^n_{k'}))$. In the case $n = 1$, the vanishing of Galois cohomologies $H^1(G, k')$ and $H^1(G, k'^*)$ implies that $X \cong \mathbb{A}^1_k$. So, a nontrivial form of $\mathbb{A}^1$, i.e., not isomorphic to $\mathbb{A}^1$, appears only if k is imperfect. In the case $n = 2$, Kambayashi [134] showed that $H^1(G, \mathrm{Aut}\,(\mathbb{A}^2_{k'})) = (1)$ and hence that all separable forms of $\mathbb{A}^2$ are trivial. The proof involves the structure theorem of $\mathrm{Aut}\,(\mathbb{A}^2_{k'})$ as an amalgamated product of $\mathrm{GL}\,(2, k')$ and the group $J_{2,k'}$ of de Jonquière transformations and a description of the Galois cohomology $H^1(G, \mathrm{Aut}\,(\mathbb{A}^2_{k'}))$ in terms of the Galois cohomologies $H^1(G, \mathrm{GL}\,(2, k'))$ and $H^1(G, J_{2,k'})$. The structure of $H^1(G, \mathrm{Aut}\,(\mathbb{A}^n_{k'}))$ is not known for $n \geq 3$ and remains as an important open problem.

We can consider a form of the additive group scheme G_a as well. Namely, a group scheme G defined over k is a *form of G_a* if $G \otimes_k \overline{k} \cong G_{a,\overline{k}}$. In the latter case, Russell [252] classified all the forms of G_a.

Theorem 1.4.2. *Let G be a form of G_a. Then G is isomorphic to a subgroup of $G_a^2 = \mathrm{Spec}\,k[x, y]$ defined by an equation $y^{p^n} = a_0x + a_1x^p + \cdots + a_m x^{p^m}$, where $a_0 \neq 0$.*

Let X be a form of $\mathbb{A}^1$. Let C be a normal complete curve defined over k such that X is an open set of C. Such C is called a *k-normal completion*. Let k'/k be a finite extension such that $X' := X \otimes_k k' \cong \mathbb{A}^1_{k'}$. Let C' be a k'-normal completion of X'. Then C' is the normalization of $C \otimes_k k'$, and is isomorphic to $\mathbb{P}^1_{k'}$. Hence $C' \setminus X'$ consists of one smooth point[9]. So, $C \setminus X$ consists of a k'-rational one-place point. The germs of research are found in Rosenlicht [250], where he considered the function fields of the forms of $\mathbb{A}^1$ whose k-normal completions have arithmetic genus > 1. The forms of $\mathbb{A}^1$ have been studied through these k-normal completions. There are several references [138, 139, 86]. We just summarize what are observed there. We let k be an imperfect field of char $(k) = p > 0$ and k_s a separable algebraic closure of k.

(1) *All forms of G_a are classified as in Theorem 1.4.2* [252].
(2) *Let C be a k-normal complete curve of arithmetic genus $g \geq 1$. Suppose that $C \otimes_k k_s$ has the group of automorphisms (leaving a one point fixed if $g = 1$) infinite. Then C has a unique singular point P_∞ such that the residue field of P_∞ is purely inseparable over k and $X := C - \{P_\infty\}$ is a principal homogeneous space for a form of G_a* [250, 252].
(3) *Let X be a form of $\mathbb{A}^1$ and let C be a k-normal completion of X. Let $\mathcal{P}ic^{\,0}_{C/k}$ be the connected component of the Picard group scheme of C.*[10] *Suppose that X has a k-rational point P_0. Then $\mathcal{P}ic^{\,0}_{C/k}$ is a commutative unipotent group, which is a form of a product of Witt vector groups of finite length. There is a closed immersion $\iota : X \hookrightarrow \mathcal{P}ic^{\,0}_{C/k}$ such that $\iota(Q) = Q - P_0$ for any field extention k'/k and any k'-rational point Q of X* ([138]; see also [139] for a detailed study of $\mathcal{P}ic^{\,0}_{C/k}$).
(4) *The forms of $\mathbb{A}^1$ are explicitly determined with additional assumptions on the existence of k-rational points in the case of arithmetic genus $g = 0, 1$ and the case where C is hyperelliptic* [139].
(5) *The structure of the automorphism group* $\mathrm{Aut}\,(X)$ *is determined when k is separably closed. The quotient of X by a subgroup of* $\mathrm{Aut}\,(X)$ *is again a form of $\mathbb{A}^1$* (see [139]).
(6) *We say that a form X of $\mathbb{A}^1$ is* of simple type *if $X \otimes_k k' \cong \mathbb{A}^1_{k'}$ for a simple purely inseparable extension k'/k of exponent one, i.e., $k' = k(\lambda)$ with $\lambda \notin k$ and $\lambda^p \in k$. A natural number g is the arithmetic genus of a certain form of $\mathbb{A}^1$ of simple type if and*

[9]Here a point on a curve is smooth if and only if the point is geometrically normal.

[10]If tensored with $\overline{k}$ an algebraic closure of k, this is the generalized Jacobian variety of $C \otimes_k \overline{k}$ due to Rosenlicht.

only if $g = (p-1)(i-1)/2$ *for some integer* $i \geq 1$ *not divisible by* p [86].

Comments. (1) In considering a form of $\mathbb{A}^1$, the role of a derivation is essential. For example, let $k' = k(\lambda)$ with $\lambda^p = \alpha \in k, \lambda \notin k$. Let $A' = k'[t]$ be a polynomial ring over k' in one variable t. Let D be a k-derivation on A' defined by $D(\lambda) = 1, D^p = 0$ and $D(t) = -t^p$. Then $A := \text{Ker } D = k[t^p, t + \lambda t^p]$. Write $y = t + \lambda t^p$ and $x = t^p$. Then $A = k[x, y]/(y^p - x - \alpha x^p)$ gives a k-form of $\mathbb{A}^1$.

(2) The forms of $\mathbb{A}^1$ are not yet fully exploited. But there are still geometric applications. Suppose that k is algebraically closed field. One of the applications is a unirational surface X with a fibration $f : X \to C := \mathbb{P}^1$ whose generic fiber $X_{\mathfrak{k}}$ is a $\mathfrak{k}$-normal completion of a $\mathfrak{k}$-form of $\mathbb{A}^1$, where $\mathfrak{k} = k(t)$. The closure of the unique singular point of $X_{\mathfrak{k}}$ is the locus Γ of unique singular points of the fibers and the restriction $f\,|_{\Gamma}\colon \Gamma \to C$ is purely inseparable. We say that such a unirational surface is *of additive type*. For example, in the case where $X_{\mathfrak{k}}$ has genus 1, X is called a *quasi-elliptic* surface, and $X_{\mathfrak{k}}$ is birational to either a curve $y^3 = x + \phi(t)x^3$ (case $p = 3$) or $y^4 = x + \psi(t)x^2 + \phi(t)^2x^4$ (case $p = 2$), where $\phi(t), \psi(t) \in k[t]$. One can determine the structure of such a quasi-elliptic surface [183, 184, 116]. So, it remains as a problem to *determine the structure of unirational surfaces of additive type.*

(3) Related to the quasi-elliptic surface in (2) above, there is a class of unirational surfaces called the *Zariski surfaces*. By definition, a Zariski surface is a smooth projective surface which is birationally isomorphic to a hypersurface in $\mathbb{A}^3$ defined by an equation of the form $z^p - f(x, y) = 0$, where p is the characteristic of the ground field k and $f \in k[x, y]$. Zariski asked if such a surface is rational provided the geometric genus is zero. Blass [36] and Lang [167] gave counterexamples. Although the defining equation is simple, this class of surfaces seem to have rich geometry and have to be exploited in the future.

(4) Here k is again an algebraically closed field of char $(k) > 0$. Let C_0 be a curve on $\mathbb{A}^2$ defined by $f = 0$. Suppose that C_0 is isomorphic to $\mathbb{A}^1$. Then one can ask whether or not C_α defined by $f = \alpha$ is isomorphic to $\mathbb{A}^1$ for all $\alpha \in k$. [11] The generic irreducibility theorem of Ganong says that the generic fiber $\mathbb{A}^2_{\mathfrak{k}}$ has one place at infinity, where $\mathfrak{k} = k(f)$. Is it a $\mathfrak{k}$-form of $\mathbb{A}^1$?

(5) According to a recent paper by Asanuma [19], if $p > 2$, any k-form of $\mathbb{A}^1$ of height e has the coordinate ring $A = k[x^{p^e}, \alpha^2\beta, \alpha\beta^\lambda]$,

[11] The author was informed of this problem by T.T. Moh in 1984. As far as the author knows, the problem is still unsolved.

where $\lambda = (p^e + 1)/2$ and $\alpha, \beta \in k^{1/p^e}[x]$ satisfying the condition

$$\alpha\partial_x(\beta) + 2\beta\partial_x(\alpha) = c \in k^{1/p} \setminus \{0\}.$$

2. Fibrations by the affine spaces and the cancellation problem

Throughout this section, we assume that the ground field k is algebraically closed. Whenever we use the homology (or cohomology) groups and the fundamental groups, we tacitly assume that the ground field is the complex field $\mathbb{C}$, or we replace those by the algebraic substitutes in the sense of étale topology and the algebraic fundamental groups.

2.1. **Fibrations by the affine spaces.** A morphism $f : X \to Y$ is called a *fibration* if f is surjective and general fibers are isomorphic to a certain *smooth* algebraic variety F. If F is the affine space $\mathbb{A}^n$, f is called an $\mathbb{A}^n$-fibration. If F is $\mathbb{A}^1_* := \mathbb{A}^1 - \{\text{one point}\}$, then f is an $\mathbb{A}^1_*$-fibration. As for the local triviality of the fibrations in the sense of Zariski topology, we can avail ourselves of the following results, which are due to a pioneering work of Kambayashi-the author [140] in the case $F = \mathbb{A}^1$ and Sathaye [264] in the case $F = \mathbb{A}^2$. There is an improvement in the case $F = \mathbb{A}^1$ by Kambayashi-Wright [143]. See also [20, 132].

Lemma 2.1.1. *With the above notations, we assume that $F = \mathbb{A}^1$ or $\mathbb{A}^2$ and that the generic fiber $X_{\mathfrak{k}}$ is isomorphic to $\mathbb{A}^1_{\mathfrak{k}}$ or $\mathbb{A}^2_{\mathfrak{k}}$ over the function field $\mathfrak{k}$ of Y. In the case $F = \mathbb{A}^2$ we assume that Y is a curve. Then there exists a dense open set U of Y such that $f^{-1}(U) \cong U \times F$.*

In fact, we choose U so that U is smooth, $f\mid_{f^{-1}(U)}: f^{-1}(U) \to U$ is affine and faithfully flat, and all the fibers of f are geometrically integral. We are then in a position to apply the results in the above-mentioned articles to obtain Lemma 2.1.1. We call an open set of the form $U \times \mathbb{A}^n$ an $\mathbb{A}^n$-*cylinder* or simply a *cylinderlike open set*. *If* $\text{char}\,(k) = 0$*, the generic fiber $X_{\mathfrak{k}}$ of an $\mathbb{A}^1$-fibration $f : X \to Y$ is necessarily isomorphic to* $\mathbb{A}^1_{\mathfrak{k}}$ (see [140, Theorem 2]). We have the following result which holds true in the case $\text{char}\,(k) > 0$ as well.

Lemma 2.1.2. *Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} A_0$ and let $f : X \to Y$ be an $\mathbb{A}^1$-fibration. Then the following conditions are equivalent.*

(1) *The generic fiber $X_{\mathfrak{k}}$ is isomorphic to $\mathbb{A}^1_{\mathfrak{k}}$.*
(2) *There exists a G_a-action on X such that A_0 is the ring of invariants in A and f is the quotient morphism.*

Proof. There exists an open set $U = \operatorname{Spec} A_0[a^{-1}]$ of Y such that $f^{-1}(U) \cong U \times \mathbb{A}^1$ by Lemma 2.1.1, where $a \in A_0$. Then $A[a^{-1}] = A_0[a^{-1}][u]$ with $u \in A$. Define a coaction $\varphi : A \to A \otimes k[t]$ as follows. Since the k-algebra A is finitely generated over k, write $A = k[v_1, \ldots, v_r]$. Since $v_i \in A_0[a^{-1}][u]$, write $a^{n_i}v_i = a_{i0} + a_{i1}u + \cdots + a_{im_i}u^{m_i}$ for $1 \le i \le r$, where $n_i \ge 0$ and $a_{i0}, \ldots, a_{im_i} \in A_0$. Define $\varphi : A[a^{-1}] \to A[a^{-1}] \otimes k[t]$ by $\varphi\,|_{A_0[a^{-1}]} = \mathrm{id}$ and $\varphi(u) = u + a^N t$, where $N \gg \max(n_1, \ldots, n_r)$. Then φ sends A into $A \otimes k[t]$. Hence this φ defines a G_a-action on X such that f is the quotient morphism. The converse is clear. $\square$

For an $\mathbb{A}^2$-fibration, there is a result by Kaliman-Zaidenberg [132].

Lemma 2.1.3. *Let the ground field be the complex field $\mathbb{C}$ and let $f : X \to Y$ be a dominant morphism of a smooth quasi-projective variety X to a smooth quasi-projective variety Y. Suppose that general fibers $f^{-1}(Q)$ for $Q \in Y$ are isomorphic to $\mathbb{A}^2$. Then there exists a Zariski open subset U of Y such that $f^{-1}(U) \cong U \times \mathbb{A}^2$.*

In [124] (see also [132]), Kaliman proved the following result as an application of Theorem 2.2.6 below.

Theorem 2.1.4. *Let f be a polynomial in $\mathbb{C}[x, y, z]$ such that the associated morphism $f : \mathbb{A}^3 \to \mathbb{A}^1$ defined by $P \mapsto f(P)$ has general fibers isomorphic to $\mathbb{A}^2$. Then $\mathbb{C}[x, y, z] = \mathbb{C}[f, g, h]$ with $g, h \in \mathbb{C}[x, y, z]$. In particular, all the fibers of f are isomorphic to $\mathbb{A}^2$.*

Let $f : X \to Y$ be a fibration with general fibers isomorphic to F. A scheme-theoretic fiber $f^*(Q)$ is defined as $X \otimes_Y \operatorname{Spec} \kappa(Q)$, where $\kappa(Q)$ is the residue field of $\mathcal{O}_{Y,Q}$. If $f^*(Q)$ is not isomorphic to F, we say that the fiber $f^*(Q)$ is *singular*. Otherwise, it is *smooth*. If $f^*(Q)_{\mathrm{red}} = F_1 + \cdots + F_r$ is the irreducible decomposition and if $\mathcal{O}_{f^*(Q),F_i}$ has length m_i as an Artin local ring then we write $f^*(Q) = m_1F_1 + \cdots + m_rF_r$ as cycles, where $\mathcal{O}_{f^*(Q),F_i}$ is the local ring of $f^*(Q)$ at the generic point of F_i. We call m_i the *multiplicity* of F_i. In what follows, we will be concerned with the case $\dim Y = 1$ and $F = \mathbb{A}^1$. Then $f^*(Q)$ is considered as an integral divisor. We call $f^*(Q)$ a *reducible* fiber if $r > 1$ and a *multiple* fiber of *multiplicity* m if $m := \gcd(m_1, \ldots, m_r) > 1$.

The following result of Suzuki-Zaidenberg [282, 310] which allows us to compute the topological Euler number of an affine surface with a fibration is quite useful. For a new proof, see also Gurjar [87].

Theorem 2.1.5. *Let X be a smooth affine surface with a morphism $f : X \to C$ with connected general fibers, where C is a smooth curve.*

Let F be a general fiber of f and let F_i $(1 \le s \le \ell)$ exhaust all singular fibers. Then we have the following equality of topological Euler numbers

$$e(X) = e(C) \cdot e(F) + \sum_{i=1}^{\ell} (e(F_i) - e(F)).$$

Furthermore, $e(F_i) \ge e(F)$ for all $1 \le i \le s$. If the equality holds for some i, then F is either isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1_*$ and F_i is isomorphic to F for all i if taken with reduced structures.

Comments. (1) Let X be a normal affine surface and let $f : X \to C$ be either an $\mathbb{A}^1$-fibration or an $\mathbb{A}^1_*$-fibration, where C is a smooth curve. Then X has only cyclic quotient singularity. See [187] for the case of $\mathbb{A}^1$-fibration, and the same arguments work for the case of $\mathbb{A}^1_*$-fibration. On the other hand, if V is a normal projective surface and $f : V \to B$ is a $\mathbb{P}^1$-fibration, V has only rational singularity. In fact, the exceptional locus of the minimal resolution of a singular point is a part of a degenerate fiber of a $\mathbb{P}^1$-fibration on a smooth projective surface. A more general result is stated in Flenner-Zaidenberg [67] as follows.

Let V be an algebraic variety (or a Moishezon variety) and let $P \in V$ be an isolated Cohen-Macaulay singularity. If there exists a Zariski open set U of V such that U is covered by closed rational curves not passing through the point P, then P is a rational singularity.

(2) Suppose that $f : X \to C$ is an $\mathbb{A}^1$-fibration or an $\mathbb{A}^1_*$-fibration on a smooth affine surface X. Let $f^*(Q)$ be a singular fiber of f. In the case of $\mathbb{A}^1$-fibration, $f^*(Q)_{\text{red}}$ is a disjoint sum of the components isomorphic to $\mathbb{A}^1$. In the case of $\mathbb{A}^1_*$-fibration, $f^*(Q) = \Gamma + \Delta$, Γ is $0, mC$ or $m_1C_1 + m_2C_2$ where $C \cong \mathbb{A}^1_*$, and $C_1 \cong C_2 \cong \mathbb{A}^1$ and C_1, C_2 meet transversally in one point, Δ is disjoint from Γ, and Δ_{red} is a disjoint sum of the components isomorphic to $\mathbb{A}^1$ provided $\Delta \neq 0$. See [195].

(3) $\mathbb{A}^1$-fibrations on normal rational surfaces are systemtically studied in [44, 45].

(4) Except for Sathaye's criterion [264], almost nothing definitive is known for local triviality or singularity of an $\mathbb{A}^n$-fibration with $\dim X \ge 3$. There is some recent trial by Kishimoto [151] on singularities of affine 3-folds which contain $\mathbb{A}^1$-cylinders.

(5) If there exists an $\mathbb{A}^1$-fibration (resp. $\mathbb{A}^1_*$-fibration) on a smooth algebraic surface X then $\overline{\kappa}(X) = -\infty$ (resp. $\overline{\kappa}(X) \le 1$). See [195].

We shall also summarize some treatments which are very close to the fibrations by the affine spaces but handled by more ring-theoretic approaches.

(6) Asanuma [19] proves that if R is a regular local ring, a finitely generated flat R-algebra A such that $A \otimes R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is isomorphic to a polynomial ring in n variables for every $\mathfrak{p} \in \operatorname{Spec} R$ and a certain fixed number n is stably isomorphic to a polynomial ring over R, i.e., $A \otimes_R R[X_1, \ldots, X_m] \cong R[Y_1, \ldots, Y_{n+m}]$ for some $m > 0$.

(7) In [20], Asanuma-Bhatwadekar proves the following result: *Let R be a commutative ring with* 1. *Let $R^{[n]}$ denote a polynomial ring in n variables over R. For a prime ideal $\mathfrak{p}$ of R, $\kappa(\mathfrak{p})$ denotes the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. An R-algebra A is said to be an $\mathbb{A}^r$-fibration over R if* (1) *A is finitely generated over R,* (2) *A is flat over R, and* (3) *$A \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})^{[r]}$ for every prime ideal $\mathfrak{p}$ of R.*[12] *Let R be a one-dimensional Noetherian domain containing the field $\mathbb{Q}$ of rational numbers. Let A be an $\mathbb{A}^2$-fibration over R. Then there exists $\xi \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[\xi]$.*

(8) There is a similar observation for $\mathbb{A}^1_*$-fibrations by Bhatwadekar-Dutta [32]. See also Asanuma-Bhatwadekar-Onoda [21].

(9) Let $f : X \to Y$ be an $\mathbb{A}^n$-fibration with $n \geq 3$. Does it follow that there exists an open set U of Y such that $f^{-1}(U) \cong U \times \mathbb{A}^n$? One obstacle seems to be the automorphism group of $\mathbb{A}^n$ whose structure is more complicated than in the $n = 2$ case.

We shall here summarize various results concerning generically rational polynomials. In the rest of this subsection, we assume that $\operatorname{char}(k) = 0$. A polynomial $f(x, y) \in k[x, y]$ is said to be *generically rational* if the general fibers of the morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$ defined by $P \mapsto f(P)$ are rational curves. When $\mathbb{A}^2$ is embedded into $\mathbb{P}^2$ in a natural way as the complement of the line at infinity ℓ_∞, we define a linear pencil $\Lambda(f)$ as the one generated by the curves C_α and $d\ell_\infty$, where C_α is the closure in $\mathbb{P}^2$ of the affine plane curve $\{f = \alpha\}$ with $\alpha \in k$ and d is the total degree of f with respect to the coordinates x, y. Let $\sigma : V \to \mathbb{P}^2$ be the shortest sequence of blowing-ups which eliminates the base points of $\Lambda(f)$. Let $\widetilde{\Lambda}(f)$ be the proper transform of $\Lambda(f)$ by σ and let $\varphi = \Phi_{\widetilde{\Lambda}(f)} : V \to B \cong \mathbb{P}^1$. This is a $\mathbb{P}^1$-fibration by the definition of f being a generically rational polynomial. Let D be the

[12]The definition of $\mathbb{A}^r$-fibration is different from ours in the point that only the fibers over maximal ideals are isomorphic to a polynomial ring in our definition. The difficulty, in our definition, lies in showing that the generic fiber $A \otimes_R Q(R)$ is isomorphic to a polynomial ring.

complement $V \setminus \mathbb{A}^2$ and let $S_1, \dots, S_r$ exhaust the irreducible components of D which lie horizontally to the morphism φ, i.e., $\varphi|_{S_i} : S_i \to B$ is dominant. If all of $S_1, \dots, S_r$ are cross-sections of φ, f is said to be *of simple type*. If S_1 is a multi-section, say an m-section, and the rest $S_2, \dots, S_r$ are cross-sections, f is said to be *of quasi-simple type*. The following results are known by far.

(1) *If f is of simple type and $r = 1$, f is a coordinate x up to an automorphism of $k[x, y]$.* (This is equivalent to the AMS Theorem).
(2) *If the curve $f = \alpha$ has at most three places at infinity for almost all $\alpha \in k$, i.e., if either f is of simple type with $r \leq 3$ or f is of quasi-simple type with $r = m = 2$,* [13] *all possible forms of f are classified* (Saito [259, 260]).
(3) *If f is of simple type with $r \geq 1$, all possible forms of f are classified* (the author-Sugie [201] and Neumann-Norbury [228]).
(4) *If f is of quasi-simple type with $m = 2$ and $r \geq 3$, all possible forms of f are classified* (Sasao [262]).

If f is a generically rational polynomial then $\operatorname{Spec} k[x, y] \otimes_{k[f]} k(f)$ is a smooth curve defined over the field $k(f)$ which is geometrically rational. By Tsen's theorem, it is rational over $k(f)$. Hence $k(x, y) = k(f, g)$ for some $g \in k(x, y)$. The polynomial f is then said to be a *field generator*. Conversely, if f is a field generator then it is generically rational. A field generator is *good* (resp. *bad*) if one can take g to be a polynomial in $k[x, y]$ (resp. otherwise). In [253, 254], Russell proved that (1) if f is a field generator, the curve $f = 0$ has always at most two points on the line at infinity ℓ_∞ when $\mathbb{A}^2 := \operatorname{Spec} k[x, y]$ is naturally embedded into $\mathbb{P}^2$, that (2) g is written as $g = h_1/h_2$ with $h_1, h_2 \in k[x, y]$ such that $h_1 = 0$ and $h_2 = 0$ have no common points, and that (3) there are no bad field generators if $\deg f < 21$. Cassou-Noguès [38] recently found two infinite series of bad field generators with degree increasing.

There is a very interesting approach to the AMS theorem. It was conjectured that if $f : \mathbb{A}^2 \to \mathbb{A}^1$ is a smooth morphism with all fibers irreducible then every fiber of f is isomorphic to $\mathbb{A}^1$. Identifying f with a polynomial in $k[x, y]$, we then say that f is a *smooth* polynomial. But this conjecture does not hold as shown by Artal-Bartolo, Cassou-Nogues and Luengo-Velasco [14]. In fact, for any integer $n > 0$, there is a smooth polynomial f_n of degree $6n + 4$ whose general fibers have genus n.

[13] The case $r = 1$ and $m \geq 2$ cannot occur, for otherwise $\operatorname{Pic}(\mathbb{A}^2)$ would not be zero.

2.2. **Characterizations of the affine spaces.** In his trial to solve the cancellation problem for $\mathbb{A}^2$, Ramanujam [246] obtained a remarkable result to characterize toplogically the affine plane as an algebraic variety. To state his characterization, recall the definition of normal compactification of an affine algebraic surface and fundamental group at infinity. Let X be a normal affine algebraic surface defined over the complex field $\mathbb{C}$. Then there exists a normal projective surface V such that X is a Zariski open set of V, V is smooth at every point of $V \setminus X$ and $V \setminus X$ consists of nonsingular irreducible curves meeting each other normally, i.e., two components of $V \setminus X$ meet transversally and each point of $V \setminus X$ is on at most two irreducible components. We then say that V is a *normal completion* (or *normal compactification*) and $V \setminus X$ considered as a reduced effective divisor on V is a divisor with simple normal crossings. A normal completion V of X is *minimal* if there does not exist a (-1)-component, i.e., an exceptional curve of the first kind, which is contractible without breaking the normality condition. Given such a normal completion V of X, we denote by D the divisor $V \setminus X$ and call the *divisor at infinity* of the completion. We can consider a Riemann metric on $V - \operatorname{Sing} V$ and a *tubular neighborhood* T_ε of D as a set of points of V whose distance from D is less than or equal to a given real number $\varepsilon > 0$ with some smoothifying modification near the intersection points of the components of D. Then the T_ε are diffeomorphic to each other when ε takes small enough values. The boundary ∂T_ε is considered to be an S^1-bundle over D, and the fundamental group $\pi_1(\partial T_\varepsilon)$ is independent of not only the choice of ε but also the choice of the normal completion V of X. Hence we denote $\pi_1(\partial T_\varepsilon)$ by $\pi_1^\infty(X)$ and call it *the fundamental group at infinity* of X. The homology groups $H_i(\partial T_\varepsilon; \mathbb{Z})$ are denoted by $H_{i,\infty}(X)$ and called the *i-th homology group of X at infinity.* Ramanujam [246] gives a concrete way of computing $\pi_1^\infty(X)$ in terms of the intersection data of the components of D including the self-intersection numbers. The results of Ramanujam are stated as follows.

Theorem 2.2.1. *Let X be a smooth affine algebraic surface defined over $\mathbb{C}$. Then X is isomorphic to the affine plane if and only if X is topologically contractible* [14] *and $\pi_1^\infty(X) = (1)$.*

For a normal completion V of X and the divisor at infinity D, we can associate the *dual graph* $\Gamma(D)$ by assigning a vertex to each irreducible component of D and connecting two vertices by as many

[14] By a theorem of J.H.C. Whitehead, this is equivalent to the condition that $H_i(X; \mathbb{Z}) = (0)$ for all $i > 0$ and $\pi_1(X) = (1)$.

edges as the number of intersection points. If we associate the self-intersection number of the component to the corresponding vertex, we call $\Gamma(D)$ the *weighted* dual graph. Ramanujam [246] gave also the following remarkable result as a consequence of Theorem 2.2.1.

Theorem 2.2.2. *Let V be a minimal normal completion of the affine plane and let D be the boundary divisor. Then each component of D is a smooth rational curve and the dual graph $\Gamma(D)$ is a linear chain.*

By making use of this result of Ramunujam, Morrow [216] gave a complete list of the weighted dual graphes of the boundary divisors of minimal normal completions of the affine plane. Kishimoto [149] found recently an easier way of proving Theorem 2.2.2 together with Morrow's list of the boundary graphes without using the topological arguments.

A third remarkable result in [246] is a construction of a smooth contractible affine surface X (called later the *Ramanujam surface*) such that X is not isomorphic to the affine plane and $X \times \mathbb{A}^1$ is diffeomorphic to $\mathbb{A}^3$, though $X \times \mathbb{A}^1 \not\cong \mathbb{A}^3$ which he was unable to show. This surface opens up a theory of *homology planes* (or *acyclic surfaces*, in other terms) and *exotic structures*. Ramanujam's surface turns out to be a homology plane with logarithmic Kodaira dimension 2, though $\mathbb{A}^2$ is the unique homology plane with logarithmic Kodaira dimension $-\infty$.

A main motivation of finding characterizations of the affine spaces was (and still is) a possible application to the cancellation problem which is to be discussed in this section. The following algebro-geometric characterization is due to the author [182].

Theorem 2.2.3. *Suppose that k is an algebraically closed field of arbitrary characteristic. Let A be an affine domain over k and let $X = \operatorname{Spec} A$. Then X is isomorphic to $\mathbb{A}^2$ if and only if*

(1) $A^* = k^*$.
(2) *A is factorial.*
(3) *X contains a cylinderlike open set $U \times \mathbb{A}^1$.*

In [182], it was assumed that X is smooth. It is thanks to Swan [284] that we can drop this assumption. For the simplified proof of Swan, we refer to [195]. When X is smooth, the conditions (1) and (2) are equivalent to $H^0_{\mathrm{et}}(X, G_m) = k^*$ and $H^1_{\mathrm{et}}(X, G_m) = (0)$. Though $H^2_{\mathrm{et}}(X, G_m)$ is related to the Brauer group of X, we are not aware of how to use it. This characterization worked successfully in the solution of the cancellation problem in dimension 2.

There are several attempts of characterizing $\mathbb{A}^3$ as an algebraic variety. We first mention two of them by the author [189, 192].

Theorem 2.2.4. *Let $X = \operatorname{Spec} A$ be a smooth affine threefold defined over $\mathbb{C}$. Then X is isomorphic to $\mathbb{A}^3$ if and only if*

(1) $A^* = \mathbb{C}^*$.
(2) *A is factorial, i.e.,* $\operatorname{Pic} X = (0)$.
(3) $H_3(X;\mathbb{Z}) = (0)$.
(4) *X contains a cylinderlike open set $U \times \mathbb{A}^2$ such that the complement $X \setminus (U \times \mathbb{A}^2)$ consists of nonsingular irreducible components.*

Theorem 2.2.5. *Let A be an affine domain of dimension 3 over $\mathbb{C}$ and let $X = \operatorname{Spec} A$. Then X is isomorphic to $\mathbb{A}^3$ if and only if the following conditions are satisfied.*

(1) $A^* = \mathbb{C}^*$.
(2) *A is factorial.*
(3) *X has the topological Euler number 1.*
(4) *X contains a cylinderlike open set $U \times \mathbb{A}^2$ such that $X \setminus (U \times \mathbb{A}^2)$ consists of the irreducible components which are factorial.*

Kaliman [124] improved Theorem 2.2.4 as follows.

Theorem 2.2.6. *Let $X = \operatorname{Spec} A$ be an affine algebraic variety of dimension three defined over $\mathbb{C}$. Then X is isomorphic to $\mathbb{A}^3$ if and only if*

(1) $A^* = \mathbb{C}^*$.
(2) *A is factorial.*
(3) $H_3(X;\mathbb{Z}) = (0)$.
(4) *X contains a cylinderlike open set $U \times \mathbb{A}^2$ such that each irreducible component of the complement $X \setminus (U \times \mathbb{A}^2)$ has at most isolated singularities.*

Kaliman and Zaidenberg [131] have shown that the characterization in Theorem 2.2.5 cannot be generalized to the case $\mathbb{A}^n$ with $n \geq 4$.

2.3. **Cancellation problem.** The problem has apparently many different roots and different presentations. The problem which we are going to discuss first was raised by Coleman-Enochs [40] in the following problem in ring theory.

Let A and B be rings. Suppose that $A[x_1, \ldots, x_n] \cong B[y_1, \ldots, y_n]$ holds for independent variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. Does it follow that $A \cong B$?

We say that A is *invariant* (resp. *strongly invariant*) if given any isomorphism $f : A[x_1, \ldots, x_n] \xrightarrow{\sim} B[y_1, \ldots, y_n]$ we have $A \cong B$ (resp. $B = f(A)$). If A and B are R-algebras for a ring R, we can ask the same question in the R-relative case and introduce the notions of A

being *R-invariant* or *strongly R-invariant*. First extensive treatments on this problem were made in Abhyankar-Heinzer-Eakin [3]. After this paper, there are several results published related to this problem. We just mention some of the results.

(1) There is an example of non-isomorphic commutative rings A and B by Hochster [113] which satisfy $A[t] \cong B[t]$. *Let $R = \mathbb{R}[x, y, z]$ with $x^2 + y^2 + z^2 = 1$ which is the coordinate ring of the real 2-sphere. Define an R-module homomorphism $\varphi : R^3 \to R$ by $\varphi(a, b, c) = ax + by + cz$. Then $E = \operatorname{Ker} \varphi$ is known to be a rank 2 projective R-module which is not free but is such that $E \oplus R$ is a free R-module of rank 3. Let A denote a polynomial ring in 2 variables over R and let B be the symmetric algebra of E over R. Then $A[t] \cong B[t]$, but A and B are not isomorphic.* This is a consequence of the existence of stably-isomorphic but non-isomorphic projective modules.

(2) *Let D be a factorial domain and let A be a factoriel domian over D such that A is a D-subalgebra of a polynomial ring $D[X_1, \ldots, X_n]$ and that A has transcendance degree 1 over D. Then A is a polynomial ring in one variable over D* (Abhyankar-Heinzer-Eakin [3]).

(3) *Let A and B be affine k-domains. Suppose that $A[x_1, \ldots, x_n] = B[y_1, \ldots, y_n]$ for indeterminates $x_1, \ldots, x_n$ over A and $y_1, \ldots, y_n$ over B. If $\operatorname{tr.deg}_k Q(A) = 1$ then $Q(A)$ is k-isomorphic to $Q(B)$* (Abhyankar-Heinzer-Eakin [3]). There is an extension of this result in Kang [144].

(4) *If R is an integrally closed domain and $B = R[x]$ then B is R-invariant. But $R[x]$ is not necessarily R-invariant if R is not normal* (Asanuma [15] and Hamann [109]).

(5) *Let A be a ring containing a subdomain C whose nonzero elements are nonzero divisors of A. Then A is strongly invariant if A has no nontrivial locally finite higher derivation* (the author-Nakai [200]).[15] It is also observed in this article that there are some relationships between strong invariance of an affine domain A over a field k and the non-existence of non-constant morphisms from $\mathbb{A}^1$ to $\operatorname{Spec} A$.

In geometric terms, the cancellation problem is stated as follows.

[15]This description is apparently related to the Makar-Limanov invariant which is to be treated later. It might be a good problem to look for some relationships between the strong invariance and the non-triviality of the Makar-Limanov invariant.

Let X, Y and V be algebraic varieties defined over k. Suppose that $X \times V$ is isomorphic to $Y \times V$. Does it follow that X is isomorphic to Y ?

In connection with the above ring-theoretic version, we usually take V to be the affine space $\mathbb{A}^n$. The theory of logarithmic Kodaira dimension and logarithmic invariants of algebraic variety due to Iitaka [114] fits the cancellation problem. In Iitaka's theory, the following ingredients are crucial.

(1) Given a smooth algebraic variety V, we can embed V into a smooth complete variety $\overline{V}$ in such a way that $\overline{V} \setminus V$ is a divisor with simple normal crossings. Here the resolution of singularities matters.
(2) Let $D := \overline{V} - V$ which is a reduced effective divisor. Consider the sheaf of logarithmic differentials $\Omega^1_{\overline{V}}(\log D)$ instead of $\Omega^1_{\overline{V}}$ and the log-canonical sheaf $\Omega^n_{\overline{V}}(\log D)$ insted of the canonical sheaf $\Omega^1_{\overline{V}}$, where $n = \dim \overline{V}$. Then the canonical divisor $K_{\overline{V}}$ is replaced by $K_{\overline{V}} + D$. In particular, the *logarithmic Kodaira dimension* is defined in the same fashion as the Kodaira dimension with $K_{\overline{V}}$ replaced by $K_{\overline{V}} + D$. The essential contribution of Iitaka [114] is a discovery of the fact that the logarithmic Kodaira dimension, denoted by $\overline{\kappa}(V)$, as well as other logarithmic invariants are independent of the choice of $\overline{V}$.

Let us begin with a result of Iitaka-Fujita [115]. We suppose in this subsection that the field k is algebraically closed.

Theorem 2.3.1. *Let X, Y and V be smooth [16] algebraic varieties over k. Assume that $\overline{\kappa}(Y) \geq 0$ and that $\overline{P}_{m_1,\cdots,m_n}(V) = 0$ for all $(m_1, \cdots, m_n) \neq (0, \cdots, 0)$. Then any isomorphism $F : X \times V \to Y \times V$ induces an isomorphism $f : X \to Y$ such that $p_Y \cdot F = f \cdot p_X$, where $p_X : X \times V \to X$ (resp. $p_Y : Y \times V \to Y$) is the projection.*

As a consequence of this result, it is shown that *if A and B are affine k-domains such that $A[x_1, \ldots, x_n] \cong B[x_1, \ldots, x_n]$ and that $\overline{\kappa}(\mathrm{Spec}\, A \setminus \mathrm{Sing}\,(\mathrm{Spec}\, A)) \geq 0$ then A is strongly invariant.*

In the geometric cancellation problem, assume that $Y \cong \mathbb{A}^2$ and $V \cong \mathbb{A}^n$. Let X be an affine scheme with the coordinate ring A. Then it follows that A is a regular factorial domain with $A^* = k^*$. Furthermore, it is rather easy to prove that $\overline{\kappa}(X) = -\infty$. In view of Theorem 2.2.3, the variety $V \cong \mathbb{A}^n$ can be cancelled off if we can show the existence of a cylinderlike open set in $X = \mathrm{Spec}\, A$. So, this case

[16] The original result is stated in more general set-ups.

was boiled down to show the following result of the author-Sugie [202] and Fujita [76].

Theorem 2.3.2. *Let X be a smooth affine surface defined over an algebraically closed field of characteristic zero. Then X contains a cylinderlike open set if and only if $\overline{\kappa}(X) = -\infty$.*

This theorem was generalized in the following points. Russell [255] and Sugie[276] proved that *instead of assuming X to be affine, it suffices to assume that X is connected at infinity.* Russell [255] also proved that k could have arbitrary characteristic. This remark was made also by M.P. Murthy and Kambayashi [136].

There is no purely ring-theoretic proof of the cancellation theorem which is strongly desirable as one thinks of the original form of the problem by Coleman-Enochs. As for a more topological proof, Gurjar [89] gave one which uses the ideas in [246] and the formula of Suzuki-Zaidenberg for the topological Euler characteristic of a smooth affine surface fibered over a smooth curve.[17] In dimension three, Kishimoto [154] considered the cancellation problem $X \times \mathbb{A}^1 \cong \mathbb{A}^4$ and gave an affirmative answer $X \cong \mathbb{A}^3$ under the hypothesis that X has a smooth normal projective embedding $X \hookrightarrow V$ such that the reduced boundary divisor $D := V - X$ is strictly numerically effective.

Although the cancellation theorem holds for $X = \mathbb{A}^2$, the cancellation problem has negative answers already in dimension two. The first counterexample was discovered by Danielewski [42]. His examples were not published by himself, but treated in detail by Fieseler [61] and tom Dieck [290]. Danielewski's example is an affine hypersurface $X_n = \{x^n y = z^2 - 1\}$ in $\mathbb{A}^3$ with $n \geq 1$. If $n = 1$, the surface X_1 is $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$, where Δ is the diagonal. Hence it has an $\mathbb{A}^1$-bundle structure over $\mathbb{P}^1$. But it has, at the same time, an $\mathbb{A}^1$-fibration $f : X_1 \to \mathbb{A}^1$ which has one fiber consisting of two connected components that are isomorphic to $\mathbb{A}^1$. The surface X_n is obtained from X_1 by taking a cyclic covering of degree n ramified along the locus $\{x = 0\}$, which is the unique reducible fiber of f. tom Dieck [290] generalized this example to the case $T(n, d) = \{x^n y = z^d - 1\}$, where $n \geq 1$ and $d \geq 2$. In their observations, the non-separated schemes of dimension one which parametrize the connected components of the $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ play certain important roles. Recently, Masuda-the author [175] generalized tom Dieck's example to the hypersurfaces $x^n y + z^d + a_2 z^{d-2} + \cdots + a_d x^d = 1$, where $n \geq 1, d \geq 2$ and $a_2, \ldots, a_d \in k$.

[17] In the appendix of [89], written in collaboration with C. R. Pradeep, a simple proof of the characterization of $\mathbb{C}^2$ due to Ramanujam is given.

On the other hand, Wilkens [301] also obtained a series of counterexamples. If we consider the cancellation problem in the equivariant context, i.e., with an action of algebraic group, the cancellation does not hold as easily shown by making use of the example of Schwarz (see the subsection 1.2). There is a reference on this subject by Masuda-the author [177].

2.4. **Exotic structures.** An *exotic structure* of the affine space $\mathbb{C}^n$ (or simply an exotic $\mathbb{C}^n$) [18] is a smooth algebraic variety Y of dimension n defined over $\mathbb{C}$ such that Y is not isomorphic to $\mathbb{C}^n$ but diffeomorphic to $\mathbb{R}^{2n}$. The interest in this topic in affine algebraic geometry began with Ramanujam [246]. By Theorem 2.2.1, there is no exotic $\mathbb{C}^2$. In fact, a smooth algebraic surface is isomorphic to $\mathbb{C}^2$ if it is homeomorphic to $\mathbb{C}^2$. Hence $n \geq 3$ if such exotic structures exist at all. The Ramanujam surface (see a remark after Theorem 2.2.2), say X, times the affine line $\mathbb{C}$ was the first pioneering example. The Ramanujam surface is obtained as follows. Let C_1 and C_2 be respectively a smooth conic and a cuspidal cubic on $\mathbb{P}^2$ such that $C_1 \cdot C_2 = P + 5Q$, let $\sigma : V \to \mathbb{P}^2$ be the blowing-up at the point P, let C_1' and C_2' be the proper transforms of the curves C_1 and C_2 and let $X = V - (C_1' + C_2')$. This X is the Ramanujam surface and $X \times \mathbb{A}^1$ is diffeomorphic to $\mathbb{R}^6$, though X is not isomorphic to $\mathbb{A}^2$. As pointed in the subsection 2.2, Ramanujam did this work in considering the cancellation problem for $\mathbb{A}^2$.

In dimension $n \geq 3$, the h-cobordism theory implies that a smooth affine algebraic variety of dimension n is diffeomorphic to $\mathbb{R}^{2n}$ if and only if X is contractible [51]. So, $X \times \mathbb{A}^1$ with the Ramanujam surface X is the first example of an exotic $\mathbb{C}^3$.

The topic of generalizing the Ramanujam surface was dormant for almost twenty years. The topic was taken anew by Gurjar-the author [90]. If one can find a complex algebraic surface X such that $X \not\cong \mathbb{A}^2$ but X is topologically contractible, then $X \times \mathbb{A}^1 \approx \mathbb{R}^6$ by the h-cobordism theory. By a theorem of J.H.C. Whitehead, a complex smooth algebraic surface X is contractible if $H_i(X;\mathbb{Z}) = (0)$ for every $i > 0$ and $\pi_1(X) = (1)$. We call a complex smooth algebraic surface X a *homology plane* if $H_i(X;\mathbb{Z}) = (0)$ for every $i > 0$ and a *contractible surface* if $\pi_1(X) = (1)$ further. Hence if one can classify all homology planes or at least find more examples, we are able to construct exotic $\mathbb{C}^3$'s of Ramanujam type. Along this line, all homology planes of logarithmic Kodaira dimension $-\infty$ or 1 are classified in [90]. Homology planes or more generally *log* $\mathbb{Q}$*-homology planes* will form a class of open surfaces which provides us with abundant examples that we can

[18] Here we denote $\mathbb{A}^n$ by $\mathbb{C}^n$ to suggest applications to the analytic case.

use as the test cases in considering various problems. We shall discuss this topic in the subsequent sections.

There are various trials to construct exotic $\mathbb{C}^n$'s which are not of Ramanujam type or product type. For example, Kaliman [122] shows that a hypersurface $x+x^k y^m+y^l z^{k-1}+t^n=0$ is diffeomorphic to $\mathbb{R}^6$ but algebraically not equivalent to $\mathbb{C}^3$ provided that $k \geq 4$, $(k-1,m)=1$, $(m(k-1),n)=1$, $n>k$ and $\gcd(m,l)>(k-1)^2 n$. In [39], Choudary-Dimca introduced a family of hypersurfaces $X_{d,a}$ in $\mathbb{C}^{2m}$ which are diffeomorphic to $\mathbb{C}^{2m-1}$, where d and a are discrete parameters. In [309], Zaidenberg discussed various aspects of exotic structures on the affine space $\mathbb{C}^n$, which includes (1) construction of exotic $\mathbb{C}^n$'s by taking the products of smooth contractible varieties and invariants with which one can distinguish exotic $\mathbb{C}^n$'s from each other, (2) analytically exotic $\mathbb{C}^n$'s, where X is an analytically exotic $\mathbb{C}^n$ if it is an exotic $\mathbb{C}^n$ and not biholomorphic to $\mathbb{C}^n$, (3) deformation and modification of exotic $\mathbb{C}^n$'s, (4) examples of affine hypersurfaces which are exotic $\mathbb{C}^n$'s, (5) construction of contractible threefolds with $\mathbb{C}^*$-actions due to Koras and Russell and Makar-Limanov invariant.

The last topic is particularly interesting. In their proof of the lineaizability of hyperbolic torus actions on $\mathbb{A}^3$, Koras-Russell introduced a family of contractible threefolds, e.g., $x+x^2y+z^2+t^3=0$ which is called the *Russell cubic*, and the exoticness of these threefolds was the key of proving the linearizability [160]. This was discussed and proved successfully by Makar-Limanov [170] and Kaliman-Makar-Limanov [127] by means of the *Makar-Limanov invariant* which measures the abundance of the G_a-actions on algebraic varieties.

In [310], the following result is shown. *Let X and Y be smooth, irreducible varieties of hyperbolic type. If for some k, m there is a biholomorphic map $X \times \mathbb{C}^k \to Y \times \mathbb{C}^m$, then X and Y are biholomorphic.* Furthermore, *for any $n \geq 3$, there exists a countable set of affine algebraic structures of $\mathbb{R}^{2n}$ which are pairwise biholomorphically nonequivalent.*

We finally add that there is a very nice survey article by Zaidenberg [311] on exotic structures.

3. Open algebraic surfaces and threefolds

We have seen various geometric approaches employed to solve the problems which can be stated in purely algebraic terms. One example is the cancellation problem for $\mathbb{A}^2$. It is, indeed, a honest impression of the author that algebra, geometry and topology can get well-mixed together when treating the problems in affine algebraic geometry. An *open* algebraic variety (or a *non-complete* algebraic variety) is simply an algebraic variety which is not necessarily complete. Hence complete algebraic varieties are contained in the class. By Nagata [220], any algebraic variety X can be embedded as an open set into a complete algebraic variety $\overline{X}$. In the case of $\mathrm{char}\,(k) = 0$, by Hironaka's weak theorem on resolution of singularities, one can assume that $\overline{X}$ is smooth and $\overline{X} \setminus X$ is a divisor with simple normal crossings. Let D be the reduced effective divisor such that $\mathrm{Supp}\, D = \overline{X} \setminus X$. By Iitaka [114] and others, it was realized that one can handle more efficiently geometric properties of X by looking a pair $(\overline{X}, D)$ instead of X itself. Thus one can apply various geometric tools on the complete algebraic varieties or the projective algebraic varieties. In what follows, we mainly consider the case of algebraic surfaces. It is partly because of the lack of the author's knowledge on higher-dimensional algebraic varieties. In this section, k is an algebraically closed field.

3.1. **Almost minimal models.** In the theory of projective algebraic surfaces, it is important to find minimal models in the birational classes of the given algebraic surfaces. If one tries to classify open algebraic surfaces, it is necessary to find a suitable *minimal model* of a given open algebraic surface. We discuss the *smooth* case first and then the case with singular points admitted which come into sight in finding the minimal models.

Let X be a smooth algebraic surface and let V be a smooth complete surface such that X is an open set of V and $D := V \setminus X$ (or simply $D = V - X$) is a divisor with simple normal crossings. We say that (V, D) is a *smooth normal completion* (or *compactification*) of X. Note that a smooth complete surface is necessarily projective. In the projective case, the process of obtaining a minimal model is to find an irreducible curve C on V such that $(K_V \cdot C) < 0$ and $(C^2) < 0$, which is a smooth rational curve with self-intersection number -1 (simply a (-1) *curve*), where K_V is the canonical divisor of V. Castelnuovo's criterion then asserts that if C is a (-1) curve on V there exists a birational morphism $\sigma : V \to W$ such that $\sigma(C)$ is a smooth point of W and σ induces an isomorphism between $V \setminus C$ and $W \setminus \{\sigma(C)\}$. In the case of open algebraic surfaces, we do the same things for $D + K_V$ instead of K_V.

Let (V, D) be a pair of a smooth projective surface and a reduced effective divisor with simple normal crossings and let $D = D_1 + \cdots + D_n$ be the irreducible decomposition. An admissible rational twig is a connected partial sum T of D such that the dual graph $\Gamma(T)$ is a linear chain of $\Gamma(D)$ connected to $\Gamma(D \setminus T)$ at only one end component of $\Gamma(T)$ and every component D_i of T is a rational curve with $({D_i}^2) \leq -2$. An admissible rational maximal twig is the one with maximal number of components. Similarly, we define an admissible rational rod R (resp. fork F) to be a connected component of D such that the dual graph $\Gamma(R)$ (resp. $\Gamma(F)$) is a linear chain of rational curves (resp. the exceptional resolution graph of a quotient singularity) and every component D_i of R (resp. F) has $({D_i}^2) \leq -2$. We set $\{T_\lambda\}, \{R_\mu\}$ and $\{F_\nu\}$ be the set of all admissible rational maximal twigs, rods and forks. By the convention, we take $\{T_\lambda\}, \{R_\mu\}$ and $\{F_\nu\}$ to be mutually disjoint. Whenever $G = D_1 + \cdots + D_s$ is one of $\{T_\lambda\}, \{R_\mu\}$ and $\{F_\nu\}$, we determine a $\mathbb{Q}$-divisor $\mathrm{Bk}\,(G) := \sum_{j=1}^{s} \beta_j D_j$ by

$$(D + K_V - \mathrm{Bk}\,(G) \cdot D_j) = 0 \quad \text{for } 1 \leq j \leq s.$$

Since the intersection matrix of G is negative definite, the $\mathbb{Q}$-divisor $\mathrm{Bk}\,(G)$ is then an effective divisor with $\mathrm{Supp}\,\mathrm{Bk}\,(G) = \mathrm{Supp}\,G$.

Let

$$D^{\#} = D - \sum_\lambda \mathrm{Bk}\,(T_\lambda) - \sum_\mu \mathrm{Bk}\,(R_\mu) - \sum_\nu \mathrm{Bk}\,(F_\nu).$$

Then it holds that:

(1) $D^{\#}$ is an effective divisor such that $\mathrm{Supp}\,D^{\#}$ and $\mathrm{Supp}\,D$ differ from each other by a disjoint union of rods and forks whose components are (-2) curves, where a $(-m)$ curve is a smooth rational curve with self-intersection number $-m$.
(2) $\mathrm{Bk}\,D$ has negative definite intersection matrix.
(3) $(D^{\#} + K_V \cdot Y) \geq 0$ for every irreducible component Y of D except for the irrelevant components which we can describe explicitly (cf. [206, 208] for the definition and related results).

If E is an irreducible curve on V such that $(D^{\#} + K_V \cdot E) < 0$, E is not a component of D, and the intersection matrix of $E + \mathrm{Bk}\,D$ is negative definite, then E is a (-1) curve meeting at most two irreducible components of $\mathrm{Bk}\,D$. Let $f : V \to \overline{V}$ be a composite of the contractions of E and all consecutively contractible components of $\mathrm{Bk}\,D$. Then $\overline{D} := f_*D$ is a reduced effective divisor with simple normal crossings, each connected component of $f(\mathrm{Supp}\,(\mathrm{Bk}\,(D)))$ is an admissible rational twig, rod or fork if it is not a point, and $f_*(\mathrm{Bk}\,(D)) \leq \mathrm{Bk}\,(\overline{D})$. On the other hand, there might exist an irreducible component E of D, called a *superfluous component*, such that E is a (-1) curve, E meets at most two components of D, $(D^{\#} + K_V \cdot E) < 0$, and the intersection matrix of $E + \mathrm{Bk}\,(D)$ is negative definite. In this case too, let $f : V \to \overline{V}$ be a composite of the contractions of E and all consecutively contractible components of $\mathrm{Bk}\,(D)$. The image $\overline{D} := f_*(D)$ is a divisor with simple normal crossings. Applying these two kinds of contractions, we can reach to a pair, which we denote by $(\widetilde{V}, \widetilde{D})$, satisfying either one of the following two conditions for every irreducible curve C on V:

(i) $(\widetilde{D}^{\#} + K_{\widetilde{V}} \cdot C) \geq 0$.
(ii) $(\widetilde{D}^{\#} + K_{\widetilde{V}} \cdot C) < 0$ and the intersection form of $C + \mathrm{Bk}\,(\widetilde{D})$ is not negative definite.

We call such a pair an *almost minimal model* (or *relatively minimal model*). Starting with a pair (V, D), we can thus reach to an almost minimal model by applying the contractions which are described as above.

The process of removing $\mathrm{Bk}\,(D)$ is called the *peeling* of the *bark*. The idea of obtaining an almost minimal model of a pair by the peeling of the barks and the contractions of irrelevant or superfluous components was originated in Kawamata [146] and fully developed in the author-Tsunoda [206] (see also [188]). A main point is the invariance $h^0(V, n(D+K_V)) = h^0(\widetilde{V}, n(\widetilde{D}+K_{\widetilde{V}}))$ for every $n \geq 0$. Hence it follows that $\overline{\kappa}(X) = \overline{\kappa}(\widetilde{X})$, where $X = V - D$ and $\widetilde{X} = \widetilde{V} - \widetilde{D}$. Furthermore, it is shown that X is *affine-ruled*, i.e, X contains a cylinderlike open set, if and only if so is $\widetilde{X}$. Furthermore, we can pass to a normal projective surface $\overline{V}$ by contracting the connected components of $\mathrm{Bk}\,(D)$. The rods and the forks are contracted to the isolated quotient singularities on $\overline{V}$, whereas the maximal twigs are contracted to the cyclic quotient singularities on the boundary divisor $\overline{D}$ which is the direct image of D on $\overline{V}$.

Comment. The last remark on the contraction to a normal projective surface $\overline{V}$ suggests that we can start the arguments with a pair $(\overline{V}, \overline{\Delta})$, where $\overline{V}$ is a normal projective surface and $\overline{\Delta}$ is a reduced effective Weil divisor on $\overline{V}$. The pair $(\overline{V}, \overline{\Delta})$ is said to have *log terminal singularities* when

(1) $\overline{\Delta} + K_{\overline{V}}$ is a $\mathbb{Q}$-Cartier divisor;
(2) If $f : V \to \overline{V}$ is the minimal resolution of singularities, then the proper transform Δ of $\overline{\Delta}$ is a divisor with simple normal crossings and

$$\Delta + K_V = f^*(\overline{\Delta} + K_{\overline{V}}) + \sum_{j=1}^{n} a_j E_j$$

with $a_j \in \mathbb{Q}$ and $-1 < a_j \leq 0$, where $\{E_j\}_{1 \leq j \leq n}$ is the set of irreducible exceptional curves of f.

It then turns out that $(\overline{V}, \overline{\Delta})$ has log terminal singularities if and only if $D := \Delta + \sum_{j=1}^{n} E_j$ is a divisor with simple normal crossings, $\overline{V}$ has only quotient singularities, and the exceptional curves of the resolution of a singular point on $\overline{\Delta}$ (if it exists) is an admissible rational twig meeting Δ in a single point. We call the pair $(\overline{V}, \overline{\Delta})$ a *log projective surface* and $f : (V, D) \to (\overline{V}, \overline{\Delta})$ the *minimal resolution*. Then we can develop the theory of relatively minimal models of log projective surfaces along with the theory of peeling to obtain almost minimal models. An irreducible curve $\overline{C}$ on $\overline{V}$ is a *log exceptional curve of the first kind* (resp. *the second kind*) if $(\overline{\Delta} + K_{\overline{V}} \cdot \overline{C}) < 0$ and $(\overline{C}^2) < 0$ (resp. $(\overline{\Delta} + K_{\overline{V}} \cdot \overline{C}) = 0$ and $(\overline{C}^2) < 0$). With the minimal resolution $f : (V, D) \to (\overline{V}, \overline{\Delta})$ as above, the pair (V, D) is almost minimal if and only if $(\overline{V}, \overline{\Delta})$ is relatively minimal, i.e., there are no log exceptional curves of the first kind. See [195] for the details.

3.2. **Classification of open algebraic surfaces.** Let (V, D) be as in the previous subsection, which is not necessarily almost minimal. The *logarithmic Kodaira dimension* of the pair $\overline{\kappa}(V, D)$ (or simply the *Kodaira dimension*) is, by definition, $-\infty$ if $h^0(V, n(D+K_V)) = 0$ for every $n > 0$ and $\max_{n \in \mathbb{L}} \dim \Phi_n(V)$, where $\mathbb{L} := \{n \in \mathbb{N}; |n(D + K_V)| \neq \emptyset\}$ and Φ_n stands for the rational mapping $V \cdots\to \mathbb{P}^{\dim |n(D+K_V)|}$ associated with the linear system $|n(D + K_V)|$ for $n \in \mathbb{L}$. When we are interested in $X := V - D$, we write $\overline{\kappa}(X)$ instead of $\kappa(V, D)$ and call it the *logarithmic Kodaira dimension* of X.

Now suppose that the pair (V, D) is almost minimal. Then $\overline{\kappa}(V, D) \geq 0$ if and only if $D^{\#} + K_V$ is numerically effective, i.e., $(D^{\#} + K_V \cdot C) \geq 0$ for every irreducible curve C on V. In this case, $D^{\#} + K_V$ is the numerically effective part in the Zariski-Fujita decomposition

$$D + K_V = (D + K_V)^{+} + (D + K_V)^{-}.$$

(See Fujita [76].) In fact, Kawamata [146] clarified the concrete process of obtaining the numerically effective part $(D + K_V)^{+}$. When $D^{\#} + K_V$ is not numerically effective, we can make use of the cone theorem of Mori [215] to obtain the following result (see the author-Tsunoda [206, 207]).

Theorem 3.2.1. *Suppose that* (V, D) *is almost minimal and* $\overline{\kappa}(D, V) = -\infty$. *Then one of the following cases takes place.*

(1) $V - D$ *is affine-ruled.*
(2) *Let* $\mathrm{Supp}\,(\mathrm{Bk}\, D) = \cup_{i=1}^{r} D_i$. *Then* $D_0 = D - \sum_{i=1}^{r} D_i$ *is irreducible and the connected component of* D *containing* D_0 *is a*

non-admissible rational fork for which D_0 is the central component (see [206] for the definition). *If* $\mathrm{char}\,(k) = 0$, *$X := V - D$ is a Platonic $\mathbb{A}^1_*$-fiber space. If $k = \mathbb{C}$ further, then $\mathbb{A}^2 \setminus \{0\}$ is the universal covering space of X, $G := \pi_1(X)$ is a small finite subgroup of* $\mathrm{GL}\,(2, \mathbb{C})$ *which is not a cyclic group, and X is isomorphic to the quotient variety $\mathbb{A}^2/G$ with the unique singular point deleted off.*

(3) $\mathrm{Supp}\,(D) = \mathrm{Supp}\,(\mathrm{Bk}\,(D))$ *and $\overline{V}$ is a log del Pezzo surface of rank one, i.e., $-(\overline{D} + K_{\overline{V}})$ is ample and* $\mathrm{rank}\,\mathrm{Pic}\,(\overline{V}) \otimes \mathbb{Q} = 1$.

It is particularly interesting that Platonic $\mathbb{A}^1_*$-fiber spaces appear here as one class of almost minimal open algebraic surfaces with log Kodaira dimension $-\infty$. To the author, it seems that geometry and topology of Platonic $\mathbb{A}^1_*$-fiber spaces are yet to be fully explored. One of the important results is stated as follows. There is a significant improvement of the result in Keel-McKernan [147].

Theorem 3.2.2. *Suppose* $\mathrm{char}\,(k) = 0$. *Let X be a smooth rational surface defined over k with $\overline{\kappa}(X) = -\infty$. Assume that X is not affine-ruled and that, for a smooth normal completion (V, D) of X, the intersection matrix of D is not negative definite. Then X has an $\mathbb{A}^1_*$-fibration over $\mathbb{P}^1$. Furthermore, X is affine-uniruled. Namely, there exists a dominant quasi-finite morphism $\pi : U \times \mathbb{A}^1 \to X$, where U is an affine curve.*

We shall mention three of the characterizations of the quotient surface $\mathbb{A}^2/G$ with G a small finite subgroup of $\mathrm{GL}\,(2, k)$. The first one is due to Gurjar-Shastri [101], the second one is due to the author [190] and Gurjar-Shastri [102] and the third one is due to Koras-Russell [162].

Theorem 3.2.3. *Let X be a normal affine surface defined over $\mathbb{C}$. Then X is isomorphic to $\mathbb{A}^2/G$ where G is a small finite subgroup of* $\mathrm{GL}\,(2, \mathbb{C})$ *if and only if X is topologically contractible and $\pi_1^\infty(X)$ is a finite group.*

Theorem 3.2.4. *Let A be a normal cofinite subalgebra of a polynomial ring in $\mathbb{C}[x_1, x_2]$, i.e., $\mathbb{C}[x_1, x_2]$ is a finite A-module, and let $X =$* $\mathrm{Spec}\,A$. *Then $X \cong \mathbb{A}^2$ if X is smooth, and $X \cong \mathbb{A}^2/G$ with a small finite subgroup G of* $\mathrm{GL}\,(2, \mathbb{C})$ *if X is singular. In the singular case, A is factorial if and only if X is isomorphic to a hypersurface $x^2 + y^3 + z^5 = 0$ in $\mathbb{A}^3$. Furthermore, X is affine-ruled if and only if G is cyclic.*

Theorem 3.2.5. *Let X be a normal affine surface with only quotient singularities. Then X is isomorphic to $\mathbb{A}^2/G$ if and only if X is topologically contractible and $\overline{\kappa}(X) = -\infty$.*[19]

We can prove Theorem 3.2.4 by making use of Theorem 3.2.2. The fact that the quotient surface $\mathbb{A}^3//G_a \cong \mathbb{A}^2$ (cf. Theorem 1.2.3) is also proved with the help of Theorem 3.2.1.

The structure of del Pezzo surfaces are well understood in the smooth case. The Gorenstein case, i.e., the case where a log del Pezzo surface admits at most Gorenstein singularities, was observed by Demazure [49], Hidaka-Watanabe [112], Furushima [78], the author-Zhang [210, 211, 212] and Ye [305]. In the general case, there are several interesting results obtained on log del Pezzo surfaces of rank one. One is about when it is isomorphic to $\mathbb{P}^2/G$ with a finite group G (see Zhang [313] and Gurjar-Pradeep-Zhang [100]) and another is the finiteness of π_1 for its smooth part by Gurjar-Zhang [105, 106] (see [74] for an analytic proof). Further, there is a work of Alekseev [9] on the fractional index of $-K_{\overline{V}}$.

In order to state the results in the case of logarithmic Kodaira dimension ≥ 0, we tacitly assume that there are no log exceptional curves of the second kind which lie outside of $\operatorname{Supp} \overline{\Delta}$.

Lemma 3.2.6. *Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution. If $\overline{\kappa}(V, D) = 0$ or 1, then $(D^\# + K_V)^2 = 0$. If $\overline{\kappa}(V, D) = 2$, then $(D^\# + K_V)^2 > 0$. Furthermore, $D^\# + K_V$ is a nef $\mathbb{Q}$-divisor. If $D^\# + K_V \equiv 0$ then $n(D^\# + K_V) \sim 0$ for some positive integer n. If $D^\# + K_V \not\equiv 0$ and $(D^\# + K_V)^2 = 0$ then $\kappa(V, D) = 1$.*

There is a structure theorem of Kawamata [146, 188, 195] in the case $\overline{\kappa}(V, D) = 1$.

Theorem 3.2.7. *Let $(\overline{V}, \overline{\Delta})$ be a relatively minimal log projective surface and let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution. We assume that there are no (-1) curves E on V such that $(D^\# + K_V \cdot E) = 0$;* we can assume this condition satisfied by contracting such (-1) curves if there exist any. *Suppose that $\overline{\kappa}(V, D) = 1$. Then $n(D^\# + K_V)$ is a Cartier divisor and $|n(D^\# + K_V)|$ is composed of an irreducible pencil Λ without base points for some positive integer n. Then the pencil Λ defines either an elliptic fibration or an $\mathbb{A}^1_*$-fibration on $V - D$. One can*

[19] Here $\overline{\kappa}(X)$ signifies the logarithmic Kodaira dimension of a minimal resolution $\widetilde{X}$ of singularities of X and it is not necessarily equal to $\overline{\kappa}(X^\circ)$, where X° is the smooth part of X.

write down the log canonical divisor formula which expresses $D^{\#}+K_V$ in terms of the lift of a divisor on the base curve and the fiber components (see [195] for the details).

Comments. (1) By Fujita [77], we can classify the relatively minimal log projective surfaces $(\overline{V},\overline{\Delta})$ when $\overline{\kappa}(V,D)=0$.

(2) With the notations and assumptions as above, we can consider the log pluri-canonical mapping

$$\overline{\Phi}:\overline{V}\to\overline{V}_c:=\mathrm{Proj}\,(\bigoplus_{n\geq 0}H^0(V,n(D+K_V)))$$

when $\overline{\kappa}(V,D)=2$. Let $\overline{\Gamma}_c$ be the direct image of $\overline{\Delta}$ by $\overline{\Phi}$. Then $(\overline{V}_c,\overline{\Gamma}_c)$ has only log canonical singularities (see [195] for the definition and the relevant results). Let $\mathrm{LCS}(\overline{V}_c,\overline{\Gamma}_c)$ be the set of all log canonical singular points which are not quotient singular points. Let

$$\overline{V}_{c,0}:=\overline{V}_c-\overline{\Gamma}_c-\mathrm{LCS}(\overline{V}_c,\overline{\Gamma}_c).$$

Then the following analogue of Miyaoka-Yau inequality in the projective case holds:

$$((D^{\#}+K_V)^2)\leq 3\left\{e(\overline{V}_{c,0})+\sum_P\left(\frac{1}{|G_P|}-1\right)\right\},$$

where $e(\overline{V}_{c,0})$ is the topological Euler number and $|G_P|$ is the order of the local fundamental group G_P of a germ $(\overline{V}_c,P)$ with P ranging over the set of quotient singular points of $\overline{V}_c$. (See Kobayashi [155] and the author-Tsunoda [209].)

3.3. **Homology planes.** A complex smooth algebraic surface X is said to be a *homology plane* (resp. $\mathbb{Q}$-*homology plane*) if $H_i(X;\mathbb{Z})=(0)$ (resp. $H_i(X;\mathbb{Q})=(0)$) for every $i>0$. If X admists quotient singular points, we call it a *log homology plane* or *log* $\mathbb{Q}$-*homology plane*. A log $\mathbb{Q}$-homology plane is affine by Fujita [77]. A normal affine surface admitting at most quotient singularities is a log homology plane if it is topologically contractible. The converse is true if $\pi_1(X)=(1)$ by a theorem of J.H.C. Whitehead.

As explained earlier, the first example of a homology plane which is not isomorphic to $\mathbb{A}^2$ but contractible was constructed by Ramanujam [246] and its construction is given in the subsection 2.4. The Ramanujam surface has log Kodaira dimension 2 and its fundamental group at infinity $\pi_1^\infty(X)$ is not trivial. A big breakthrough was brought on the study of homology planes by the proof of its rationality due to Gurjar-Shastri [103, 104] (the case of homology planes), Pradeep-Shastri [244] and Gurjar-Pradeep-Shastri [99] (the case of $\mathbb{Q}$-homology planes) and

Gurjar-Pradeep [98] (the case of log $\mathbb{Q}$-homology planes). The advantage of the proof of rationality is that we have only to consider rational surfaces in search of homology planes. The rationality was known in [90] when $\overline{\kappa} \leq 1$. In the last case, a surface has an $\mathbb{A}^1$-fibration or an $\mathbb{A}^1_*$-fibration, and one can compute the homology groups and the fundamental group of such a surface with $\mathbb{A}^1$-fibration or $\mathbb{A}^1_*$-fibration. The computation has led to a complete determination of homology planes and contractible surfaces with $\overline{\kappa} = 1$. It is rather easy to show that *the affine plane* $\mathbb{A}^2$ *is a unique homology plane with* $\overline{\kappa} = -\infty$ *and there are no homology planes with* $\overline{\kappa} = 0$. This result does not hold for $\mathbb{Q}$-homology planes or log $\mathbb{Q}$-homology planes.

The case of $\overline{\kappa} = 1$ is taken up also by tom Dieck-Petrie [294, 293]. Denote by $V(n,a)$ the surface in $\mathbb{A}^3$ given by $((xz+1)^n - (yz+1)^a)/z = 1$, where a, n are relatively prime integers with $1 < a < n$. There is a surface $W(n,a)$ obtained from the Hirzebruch surface Σ_1 by removing a suitable reducible curve. It is shown that $V(n,a) \cong W(n,a)$ and that $V(n,a)$ is a basic contractible homology plane with $\overline{\kappa} = 1$ from which all other contractible homology planes with $\overline{\kappa} = 1$ are obtained by a simple procedure.

Let X be a homology plane and let (V,D) be a smooth normal completion of X. Since X is rational, there is a birational morphism $\sigma : V \to W$, where W is isomorphic to $\mathbb{P}^2$ or a Hirzebruch surface Σ_n. tom Dieck-Petri [295, 296] and tom Dieck [288, 289, 291] considered the cases where $\sigma(D)$ is a line arrangement (called a *plane divisor*) on $\mathbb{P}^2$ or a suitable reducible curve (called an *optimal minimal curve*) on Σ_n and succeeded in constructing many interesting homology planes, especially those with $\overline{\kappa} = 2$. There is an extensive explanation about the construction from line arrangements in [311].

Theorem 3.3.1. *Let X be a homology plane of general type, i.e., $\overline{\kappa}(X) = 2$. Then the following assertions hold.*

(1) *There exists a smooth normal completion (V,D) of X such that (V,D) is almost minimal. Hence it holds that $(D^{\#} + K_V)^2 \leq 3e(X)$.*
(2) *There are no curves on X which are homeomorphic to $\mathbb{A}^1$.*

The second assertion was first proved by Zaidenberg [310]. In the same paper, he also shows that there exists a unique contractible curve on a homology plane of $\overline{\kappa} = 1$. In [209] and [93], the author together with Tsunoda and Gurjar proved independently the assertion (2) as a consequence of the Miyaoka-Yau inequality (cf. the assertion (1) of Theorem 3.3.1). The problem concerning the existence of contractible curves on homology planes was finished with the result in the case $\overline{\kappa} = 0$ treated in Gurjar-Parmeshwaran [96].

Comments. (1) Log $\mathbb{Q}$-homology plane was first recognized in the author-Sugie [204] for the quotient space of a homology plane by a finite automorphism group. The Platonic $\mathbb{A}^1_*$-fiber spaces $\mathbb{A}^2/G$ exhaust log $\mathbb{Q}$-homology planes of $\overline{\kappa} = -\infty$ which have no $\mathbb{A}^1$-fibrations.

(2) The construction of the Ramanujam surface was pursued in Sugie [277]. Let F and G be irreducible curves on $\mathbb{P}^2$ such that (i) $\deg F = m$ and $\deg G = m+1$, (ii) F and G are homeomorphic to $\mathbb{P}^1$, and (iii) F and G meet in two smooth points R and S. Let $\pi : V_1 \to \mathbb{P}^2$ be the shortest sequence of blowing-ups with centers at R, S and the singular points of F, S so that $\pi^{-1}(F+G)$ is a divisor with simple normal crossings and let $\sigma : V_2 \to V_1$ be a sequence of blowing-ups of arbitrary length with centers at the infinitely near points of one of R and S (either one is fixed) such that the dual graph of $D := (\pi \cdot \sigma)^{-1}(F+G) - E$ is a minimal tree, i.e., any (-1) component meets at least three other components. Assume that $V_2 - D$ is a homology plane. Then it is shown that $m \leq 3$.

(3) Homology planes provide a very nice class of open algebraic surfaces of general type, i.e., $\overline{\kappa} = 2$, for which we can test the validity of various working hypothesis. Throughout the construction of the theory of open algebraic surfaces, there was a firm belief that the theory must go in parallel with the Enriques-Kodaira classification of projective surfaces even if there were some modifications to be made. One example is a homology plane of general type having an $\mathbb{A}^{**}$-fibration, where $\mathbb{A}^{**}$ is the affine line with two points punctured. An $\mathbb{A}^{**}$-fibration is considered to be a substitute in the open case of a fibration by curves of genus 2 in the projective case. Xiao [304] (see also [236]) proved that the inequality $c_1^2 \leq 2c_2$ occurs for a smooth projective surface with a relatively minimal fibration of curves of genus 2. Homology planes (even log $\mathbb{Q}$-homology planes) with $\mathbb{A}^{**}$-fibrations are observed in the author-Sugie [205]. In [279], Sugie and the author showed that the same inequality, i.e., $(D^{\#} + K_V)^2 \leq 2e(X)$, occurs indeed for a homology plane of general type having an $\mathbb{A}^{**}$-fibration. Sugie [278] extended the observations in [205] to the case of homology planes with $\mathbb{A}^{(*N)}$-fibrations and classified the singular fibers in the case $N = 3$, where $\mathbb{A}^{(*N)}$ signifies the affine line with N points punctured.

(4) Zaidenberg [308] also observed algebraic surfaces with $\mathbb{A}^{**}$-fibrations in connection with the construction of homology planes and exotic structures on $\mathbb{C}^n$, where he considers an $\mathbb{A}^{**}$-fibration such that every

irreducible component of a singular fiber is $\mathbb{A}^1_*$ and presents an explicit description of the surface by means of some simple discrete data. His observation leads to a complete determination of $\mathbb{A}^{**}$-polynomials, which are polynomials f giving rise to an $\mathbb{A}^{**}$-fibration $f : \mathbb{A}^2 \to \mathbb{A}^1$.

(5) In [64], Flenner-Zaidenberg treated the deformation theory of $\mathbb{Q}$-homology planes. If V is a smooth compactification of an open algebraic surface X and $D = V \setminus X$ is a simple normal crossing divisor, consider the deformations $(\mathcal{V}, \mathcal{D})$ of (V, D) over a base space S such that the number of components of $\mathcal{D}_s$ $(s \in S)$ is locally constant on S. If X is a $\mathbb{Q}$-homology plane, the nearby fibers $X_s = V_s - D_s$ are then $\mathbb{Q}$-homology planes, and the number of moduli of (V, D) is given by $h^1(\Theta_V(D))$, where Θ_V is the tangent bundle of V. This number is computed for the minimal compactifications of $\mathbb{Q}$-homology planes of $\overline{\kappa}(X) \le 1$ and for many examples with $\overline{\kappa}(X) = 2$. Moreover, it is shown that the deformations are unobstructed in all these cases.

(6) Homology planes will play the roles of test examples in other problems like the cancellation problem and the Jacobian conjecture. We shall discuss the results in the case of the Jacobian conjecture in section 5.

3.4. **Topics related to open algebraic surfaces.** There are many applications of the theory of open algebraic surfaces. We shall pick up several topics.

(1) Let C be a plane curve and let $X := \mathbb{P}^2 - C$. According to $\overline{\kappa}(X)$, one can hope to classify the curves C. The case $\overline{\kappa}(X) = -\infty$ is treated by the author-Sugie [203]. The case $\overline{\kappa}(X) = 0$ is by Kojima [157]. The case where C is an irreducible curve with two cusps and $\overline{\kappa}(X) = 1$ is treated in Kishimoto [154]. In the case where C is an irreducible curve with at least three cusps, we have necessarily $\overline{\kappa}(X) = 2$. Such a curve C is said to be of type (d, m) if $d = \deg C$ and m is the largest multiplicity among the singularities of C. In [65], Flenner-Zaidenberg considered the case of type $(d, d-2)$ and classified rational cuspidal curves C with a point of multiplicity $m = \deg C - 2$ and at least three singular points. It turns out that $\mathrm{Sing}\,(C)$ contains exactly three singular points and that there are $[\binom{d-1}{2}]$ projective equivalence classes of such curves for every $d = \deg C$. A similar observation is made for the case of type $(d, d-3)$ in [69]. Fenske [60] continued this direction to the case of type $(d, d-4)$. There are numerous contributions done by many people, among whom we list Yoshihara [306], Matsuoka-Sakai [179] and Sakai-Tono [261].

(2) An interest of studying plane cuspidal curves lies in $\mathbb{Q}$-homology planes. For an irreducible plane curve C, the complement $X := \mathbb{P}^2 - C$ is a $\mathbb{Q}$-homology plane if and only if C is a rational curve with at most cuspidal singular points. If C is rational and cuspidal, then $\overline{\kappa}(X) = 2$ if C has at least three cuspidal points. Zaidenberg has a conjecture that a $\mathbb{Q}$-homology plane of $\overline{\kappa} = 2$ is rigid and unobstructed, i.e., $H^1(V, \Theta_V(D)) = H^2(V, \Theta_V(D)) = (0)$ for any normal completion (V, D) of X. There are several evidences and related results in Flenner-Zaidenberg [65], Orevkov-Zaidenberg [312]. In the case of the complement of a rational cuspidal plane curve, the rigidity is equivalent to saying that any equisingular embedded deformation of C is a translate of C by a projective automorphism (projective rigidity).

(3) There are further trials of implementing the Enriques-Kodaira classification of algebraic surfaces to the case of open algebraic surfaces. There are too many publications to cover all. The author mentions a few of them. There are papers of Zhang to generalize the Enriques surfaces [314, 315]. To look into the properties of log projective surfaces of general type, Tsunoda-Zhang [287] and Zhang [316] generalized Noether's inequality.

(4) In [97], Gurjar-Parmeshwaran characterized the pairs (V, D), where V is a smooth projective surface and D is a connected curve on V, for which the topological Euler number $e(V - D) \leq 0$. In particular, it is shown that if X is a smooth affine surface with $e(X) < 0$ then X has an $\mathbb{A}^1$-fibration. This is considered as the affine version of Castelnuovo's criterion of ruledness. There is a result of Vey [299] concerning the structure of rational open algebraic surfaces with Euler number ≤ 0. See also de Jong-Steenbrink [48].

3.5. **Higher-dimensional case.** Our knowledge in the higher dimensional case is very limited, and there are numerous problems to be considered. We shall discuss some of the problems.

(1) Let us consider how to characterize the affine 3-space. For example, if a smooth affine 3-fold $X = \operatorname{Spec} A$ is given in the set-ups like $X \times \mathbb{A}^n \cong \mathbb{A}^{n+3}$ or A being a cofinite subalgebra of $k[x, y, z]$, what kind of conditions would guarantee that $X \cong \mathbb{A}^3$? In Theorems 2.2.4 and 2.2.5, the existence of a cylinderlike open set $U \times \mathbb{A}^2$ is essential. But no immediate applications are possible in the above set-ups. We say that an algebraic variety X is *affine n-ruled* if there exist a dominant morphism $f : X \to Y$ and a non-empty open set U of Y such that $f^{-1}(U) \cong U \times \mathbb{A}^n$ and f is identified with the projection $U \times \mathbb{A}^n \to U$

on the open set $f^{-1}(U)$. The affine 1-ruledness is preserved under the reduction. Namely, we have the following result [189].

Lemma 3.5.1. *Let $(\mathcal{O}, t\mathcal{O})$ be a discrete valuation ring with uniformisant t. Let K and k be the quotient field and the residue field of $\mathcal{O}$, respectively. Let A be a finitely generated algebra over $\mathcal{O}$ such that A is $\mathcal{O}$-flat and that $A \otimes_{\mathcal{O}} K$ and $A \otimes_{\mathcal{O}} k$ are integral domains. Assume further that* $\mathrm{char}\, k = 0$. *If* $\mathrm{Spec}\, A_K$ *is affine* 1*-ruled, so is* $\mathrm{Spec}\, A_k$.

The affine 2-ruledness is, however, not preserved under the reduction. Hence a charactrization of $\mathbb{A}^3$ using the affine 1-ruledness instead of the affine 2-ruledness is more desirable. We already noted that the conditions $A^* = \mathbb{C}^*$ and $\mathrm{Pic}\, X = (0)$ in Theorems 2.2.4 and 2.2.5 correspond respectively to $H^0_{\mathrm{et}}(X, G_m) = \mathbb{C}^*$ and $H^1_{\mathrm{et}}(X, G_m) = (0)$. So, if $\dim X$ becomes higher, we probably need some conditions related to the vanishing of higher étale cohomologies $H^i_{\mathrm{et}}(X, G_m)$ with $i \geq 2$ or singular cohomology (homology) groups. Furthermore, $X \times \mathbb{A}^n \cong \mathbb{A}^{n+3}$ implies $\overline{\kappa}(X) = -\infty$. But we are not aware of what the condition $\overline{\kappa} = -\infty$ implies. There are some trials by Kishimoto [151, 152, 153].

Another possibility lies in the case of log Fano varieties as in the case of log del Pezzo surfaces. A log Fano variety is a normal projective variety with log terminal singularities and the anti-canonical divisor $-K_X$ ample. Define the *Fano index* of X as the largest rational number $r(X)$ such that $-K_X \sim_{\mathbb{Q}} r(X)H$ for a Cartier divisor H. In [317], Zhang shows that $\pi_1(X^0)$ of the smooth part X^0 of X is finite if $r(X) > \dim X - 2$, trivial if $r(X) > \dim X - 1$ and an abelian group of order ≤ 9 if $r(X) = \dim X - 1$.

In the case of surfaces, the quotient surface $\mathbb{A}^2/G$ with G a finite group plays an important role in the classification of open surfaces of $\overline{\kappa} = -\infty$. In fact, in the linearization theorem of a G_m-action on $\mathbb{A}^3$, the crucial part in the proof by Koras-Russell [161] is to show that the quotient surface $\mathbb{A}^3/\!/G_m$ in the hyperboric case is isomorphic to $\mathbb{A}^2/G$, or equivalently, to show that $\mathbb{A}^3/\!/G_m - \mathrm{Sing}\,(\mathbb{A}^3/\!/G_m)$ has $\overline{\kappa} = -\infty$. One may hope that this result is generalized in higher-dimensional case. Namely, *does it hold that $\mathbb{A}^n/\!/G - \mathrm{Sing}\,(\mathbb{A}^n/\!/G)$ has $\overline{\kappa} = -\infty$ when G is a reductive algebraic group ?*

4. Automorphisms

We discussed in the section one the actions of G_a and G_m on the affine spaces. We consider here the automorphism groups of affine varieties. The ground field k is an algebraically closed field.

4.1. **Automorphisms of affine surfaces.** The automorphism group $\mathrm{Aut}(\mathbb{A}^2)$ is an amalgamated product of the affine transformation group A_2 and the de Jonquière transformation group J_2, where J_2 consists of automorphisms $\varphi : (x, y) \mapsto (x, y + f(x))$ with $f(x) \in k[x]$. In particular, $\mathrm{Aut}(X)$ acts *transitively* on $\mathbb{A}^2$.

Gizatullin [81] defines a *quasi-homogeneous* surface as a smooth affine surface X such that there is an open orbit $O := \mathrm{Aut}(X) \cdot P$ with $X \setminus O$ a finite set of points. In the same paper, Gizatullin obtains the following result.

Theorem 4.1.1. *Let X be a smooth affine surface. Then the following assertions hold.*

(1) *Suppose that there exists a smooth normal completion (V, D) of X such that D consists of rational curves and the dual graph of D is a loop. Then X is quasi-homogeneous if and only if X is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1_*$.*
(2) *Suppose that $X \not\cong \mathbb{A}^1_* \times \mathbb{A}^1_*$. Then X is quasi-homogeneous if and only if there exists a smooth normal completion (V, D) of X such that D consists of rational curves and the dual graph is a linear chain* (which is called a *zigzag* in [81]).

Comments. (1) A surface X is said to be homogeneous (resp. quasi-homogeneous) with respect to algebraic groups if there exists an algebraic group G and an action of G on X such that there exists a closed point $P \in X$ such that the G-orbit of P contains all points of X (resp. all but a finite number of points). In [82], Gizatullin proves that a smooth affine surface X is quasi-homogeneous with respect to an algebraic group if it is isomorphic to exactly one of the following: $\mathbb{A}^2$, $\mathbb{A}^1 \times \mathbb{A}^1_*$, $\mathbb{A}^1_* \times \mathbb{A}^1_*$, $\mathbb{P}^2 - C$ with a smooth conic C, and $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta_i$ if $\mathrm{char}(k) > 0$), where Δ is the diagonal (resp. Δ_i, for any non-negative integer i, is the graph of the i-th power of the Frobenius endomorphism of $\mathbb{P}^1$). Furthermore, all the above affine surfaces are in fact homogeneous with respect to algebraic groups except for the case $X \cong \mathbb{P}^2 - C$ when $\mathrm{char}(k) = 2$.

(2) In the case where X is non-smooth and $\mathrm{char}(k) = 0$, Popov [239] completed a classification of G-surfaces, which is shown to be isomorphic to the affine cone $V(n_1, \ldots, n_m)$ of the image $v_{n_1,\ldots,n_m}(\mathbb{P}^1)$ by the morphism $v_{n_1} \times \cdots \times v_{n_m} : \mathbb{P}^1 \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, where $v_n : \mathbb{P}^1 \to \mathbb{P}^n$ is the Veronese map of degree n and $n_1, \ldots, n_m \geq 2$. Furthermore, he extended this result to the 3-dimensional case in [240], according to which X is homogeneous if X is smooth, X has at most one singular point and all the possible X's are listed.

(3) In [81], Gizatullin asks whether or not any quasi-homogeneous surface is homogeneous. In [83], Gizatullin-Danilov gives an example of non-homogeneous but quasi-homogeneous surface in the case $\mathrm{char}\,(k) = p \neq 0$. Let $\sigma : V \to \mathbb{P}^2$ be the blowing-up of the point $(1, 0, 0)$ and let $X := V - \sigma'(C)$, where C is the curve defined by $X_0X_1^{p-1} - X_2^p = 0$. Then X has two orbits under $\mathrm{Aut}\,(X)$ which are the dense orbit and one fixed point $(0, 0, 1)$.

(4) When a smooth affine surface X has a smooth normal completion (V, D) with D a zigzag satisfying certain "standard" condition, Gizatullin-Danilov [84, 85] reduces $\mathrm{Aut}\,(X)$ to the symmetry group related to an oriented graph Δ_X of standard completions of X.

On the other hand, Bertin [29] observed the automorphism group of an affine surface with an $\mathbb{A}^1$-fibration.

Theorem 4.1.2. *Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to \mathbb{A}^1$ and let (V, D) be a minimal normal completion of X. Let $\Gamma(V, D)$ be the dual graph of D. Then the following assertions hold.*

(1) *$\Gamma(V, D)$ is a tree.*
(2) *$\Gamma(V, D)$ is not a linear chain if and only if the $\mathbb{A}^1$-fibration ρ is unique up to automorphisms of the base curve.*
(3) *Suppose that $\Gamma(V, D)$ is not a linear chain. Then* $\mathrm{Aut}\,(X)$ *is an increasing union of algebraic subgroups whose connected identity components are solvable groups of rank ≤ 1.*[20] *For each integer N,* $\mathrm{Aut}\,(X)$ *contains a commutative algebraic subgroup of dimension $\geq N$. Furthermore, if* $\mathrm{rank}\,(\mathrm{Aut}\,(X)) = 1$ *, then all one-dimensional tori in* $\mathrm{Aut}\,(X)$ *are conjugate to each other.*

Comment. In [30, 31], Bertin also observed the fundamental group of an affine surface X with an $\mathbb{A}^1$-fibration or an $\mathbb{A}^1_*$-fibration. In the analytic case, one can define a $\mathbb{C}$-fibration or a $\mathbb{C}^*$-fibration in the same way as in the algebraic case. He then shows that (1) a finite subgroup of $\mathrm{Aut}\,(X)$ is contained in an algebraic subgroup $\mathrm{Aut}\,(V, D)$ of $\mathrm{Aut}\,(X)$, where (V, D) is a minimal normal completion of X and $\mathrm{Aut}\,(V, D)$ is the subgroup of $\mathrm{Aut}\,(V)$ consisting of automorphisms which preserve the boundary D. (2) A classical Fuchsian group is reinterpreted as the fundamental group $\pi_1(X)$ of an algebraic $\mathbb{A}^1$-fibration $\rho : X \to C$ with suitable multiple fibers. A realization of a Fuchsian group as a finitely generated discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ is explained by constructing a suitable analytic $\mathbb{C}$-fibration $\rho : X \to C$ with $C = H/\Gamma$, H being the upper half-plane. (3) Via results of Pinkham [238], normal quasi-homogeneous surface singularities, e.g. singularities of

[20] By definition, it is the dimension of a maximal torus.

Brieskorn surfaces, can be studied through $\mathbb{C}^*$-fiber spaces X (called *singular Seifert fibrations*). In particular, the types of singularities (e.g., rational, simple-elliptic, etc.) are classified with the Kodaira dimension of $X^0 = X - \mathrm{Sing}\,(X)$.

In view of the results of Gizatullin and Bertin, the following result of Bandman-Makar-Limanov [23] is noteworthy.

Theorem 4.1.3. *Let X be a complex smooth affine surface with the coordinate ring A and let* $\mathrm{ML}\,(X)$ *denote the subring of A consisting of elements which are invariant under all possible G_a-actions on X. Suppose that* $\mathrm{ML}\,(X) = \mathbb{C}$. *Then X is quasi-homogeneous. Furthermore, X is isomorphic to a Danielewski hypersurface $\{xy = p(z)\}$ in $\mathbb{A}^3$ if and only if the canonical class K_X is trivial, where $p(z) \in \mathbb{C}[z]$ such that $p(z) = 0$ has all simple roots.*

The subring $\mathrm{ML}\,(X)$ is called the Makar-Limanov invariant and was originally denoted by $\mathrm{AK}(X)$. When $\mathrm{ML}\,(X) = \mathbb{C}$, we say that X has *trivial* Makar-Limanov invariant. The Makar-Limanov invariant is one of important probes to investigate the properties of affine surfaces with G_a-actions or $\mathbb{A}^1$-fibrations. Gurjar-the author [95] and Dubouloz [55] extended a result of Gizatullin-Bertin to the case of normal affine surfaces with trivial Makar-Limanov invariant. Namely, *a log affine surface X has trivial Makar-Limanov invariant if and only if X has a minimal normal completion V such that the dual graph of $D := V - X$ is a linear chain of rational curves and $\pi_1^\infty(X)$ is a finite group.* [21]

Here we introduce the notion of ML_i-surface for $(i = 0, 1, 2)$. A smooth affine surface X is an ML_i*-surface* if $\mathrm{ML}\,(X)$ has transcendence degree i over the ground field k. So, an ML_0-surface has trivial Makar-Limanov invariant and vice versa. An ML_1-surface has a unique $\mathbb{A}^1$-fibration with affine base curve (cf. Theorem 4.1.2, (2)). An ML_0-surface X with Picard number zero has a strong similarity to the affine plane. To wit, it is shown in Gurjar-Masuda-the author-Russell [107] that *a curve on X isomorphic to the affine line is a fiber of an $\mathbb{A}^1$-fibration* (an analogue of AMS Theorem for $\mathbb{A}^2$) and that *if $X \to Y$ is a finite morphism to a smooth surface Y then Y is an* ML_0*-surface* (an analogue of Theorem 3.2.4).

If X is a $\mathbb{Q}$-homology plane, Masuda-the author [174] shows that the universal covering space of X is isomorphic to a hypersurface $\{xy =$

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[21] A *log affine surface* is a normal affine surface with at most quotient singularities. If X has an $\mathbb{A}^1$-fibration, there exist at most cyclic quotient singularities (see Comment (1) after Theorem 2.1.5). The condition that $\pi_1^\infty(X)$ be a finite group is equivalent to saying that X is not isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1_*$.

$z^n - 1\}$ with $n = |\pi_1(X)|$ provided $\mathrm{ML}\,(X) = \mathbb{C}$. Hence X is a quotient surface of the above hypersurface by $\pi_1(X)$ which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. There is a description of X in terms of weighted projective planes in Daigle-Russell [46]. Recently, Masuda-the author [176] shows that a $\mathbb{Q}$-homology plane with torus action has the universal covering isomorphic to a hypersurface $\{x^r y = z^n - 1\}$ with $n = |\pi_1(X)|$. This is a natural extension of the above result on a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant.

By Dubouloz [56], a smooth affine surface X with an $\mathbb{A}^1$-fibration $\rho : X \to C \cong \mathbb{A}^1$ is a *generalized Danielewski surface* if all fibers are reduced and all fibers possibly except for one are irreducible. It is *ordinary* if it is a hypersurface $V_{p,n}$ of $\mathbb{A}^3$ defined by $x^n y = p(z)$ with $p(z) \in \mathbb{C}[z]$. In [171, 172], Makar-Limanov shows that *an ordinary generalized Danielewski surface $V_{p,n}$ is an ML_0-surface if and only if $n = 1$ and $\deg p(z) \geq 1$.* In [56], this result of Makar-Limanov is extended to a result on generalized Danielewski surfaces which is stated in terms of weighted rooted trees.

Algebraic varieties with algebraic torus actions (or more generally with reductive algebraic group actions) are rather well investigated. The normal linearization theorem by Bass-Haboush [26] tells us that an affine scheme $X = \operatorname{Spec} A$ with a reductive algebraic group G action carries a G-vector bundle structure over a G-stable closed subscheme X_0 if X_0 satisfies certain conditions that (1) X_0 is G-stable and contains all closed orbits, (2) there is a G-equivariant retraction $\pi : X \to X_0$ and (3) X_0 is a local complete intersection in X. As a corollary, it follows that if a reductive algebraic group G acts on $\mathbb{A}^n$ and each closed orbit is a fixed point, then $\mathbb{A}^n$ is G-equivariantly isomorphic to $(\mathbb{A}^n)^G \times \mathbb{A}^m$ for certain $m \geq 0$. By making use of this corollary, one can show in the case of a G_m-action on a smooth affine algebraic variety that the quotient morphism $X \to X/\!/G_m$ is a G_m-vector bundle if $X/\!/G_m \cong X^{G_m}$.

In [235, 234], Orlik-Wagreich classified all smooth projective surfaces with G_m-actions and described the canonical equivariant resolution of surface singularity with G_m-action.

Given a G_m-action σ on an affine variety $X = \operatorname{Spec} A$ there is a direct sum decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$ which makes A a $\mathbb{Z}$-graded algebra over the ground field $\mathbb{C}$ (see Lemma 1.1.2). An effective G_m-action σ is *elliptic* (resp. *parabolic*, or *hyperbolic*) if $A_i = 0$ for $i < 0$ and $A_0 = \mathbb{C}$ (resp. $A_i = 0$ for $i < 0$ and $A_0 \neq \mathbb{C}$, or $A_i \neq 0$ for every $i \in \mathbb{Z}$). This has a clear meaning when $\dim A = 2$ and A is normal. Namely, X has a unique fixed point through which passes the closure of a general G_m-orbit if σ is elliptic, X has the one-dimensional fixed point locus which meets the closure of a general G_m-orbit in one point transversally if σ is parabolic, and σ induces an (*untwisted*, in fact) $\mathbb{A}^1_*$-fibration on X and there are finitely many fixed points if σ is hyperbolic.

Based on the work of Orlik-Wagreich, Demazure [50], Dolgachev [52] and Pinkham [238] gave a concrete way to describe the graded ring A in terms of a curve E and a Weil divisor D on E in the form $A_k = H^0(E, \mathcal{O}_E(D^{(k)}))$ when $\dim A = 2$ and the action is elliptic or parabolic. Flenner-Zaidenberg [68] extended this construction to the hyperbolic case as well.

In the course of these developments, Rynes [258] classified all smooth affine surfaces with effective G_m-actions (especially in the case of hyperbolic fixed points). Fieseler-Kaup [62, 63] explored geometric properties and classification of normal affine surfaces with G_m-actions. Finally, Flenner-Zaidenberg [68] completed a classification when X is a normal affine surface with G_m-action. They applies this result of classification to the case where X has further a G_a-action and discusses from a new view point the problem of homogeneity and quasi-homogeneity as treated at the beginning of this subsection (see [69, 70]).

The quotient morphism $q : X \to X/\!/G_m$ in the case of a hyperbolic G_m-action on a normal affine surface X is an untwisted $\mathbb{A}^1_*$-fibration. Tanimoto [286] considers under which conditions a given $\mathbb{A}^1_*$-fibration is unique and when the $\mathbb{A}^1_*$-fibration gives rise to a hyperbolic G_m-action.

Comments. (1) Let V be a generic del Pezzo surface which is generic in the sense that when we obtain V by blowing up r $(0 \le r \le 8)$ points of $\mathbb{P}^2$ we take these r points in sufficiently general positions. Then $\mathrm{Aut}\,(V)$ was determined by Koitabashi [156]. In fact, he showed that if $r \ge 9$ then $\mathrm{Aut}\,(V) = (1)$. If V is a log del Pezzo surface there are scattered results (cf. [212]) but the automorphism group is not yet completely determined. If X is a Platonic $\mathbb{A}^1_*$-fiber space then $\mathrm{Aut}\,(X)$ is an algebraic group (cf. [173]).

(2) Determination of the automorphism group of $\mathbb{A}^3$ is a long-standing problem. Nagata [222] asked if an automorphism σ of $k[x, y, z]$ given by $\sigma(x) = x - 2y(zx+y^2) - z(zx+y^2)^2, \sigma(y) = y + z(zx+y^2)$ and $\sigma(z) = z$ (called the *Nagata automorphism*) is a tame automorphism, i.e., it is expressed as a composite of affine transformations and de Jonquière transformations. Smith [272] proved that σ is stably tame, i.e., tame after adjoining several new variables (in fact, one). Kishimoto [152] observed the Cremona transformation induced by σ and factorized it to a composite of elementary links. Finally, Shestakov-Umirbaev [270, 271]

proved by making use of Poisson algebra that the Nagata automorphism is wild, i.e., non-tame. Hence, in order to determine Aut $(\mathbb{A}^3)$, we have to introduce some ideas to control the wildness of the automorphisms. The notion of stable tameness might be one idea. There are several approaches, e.g., Kishimoto [152], Edo [58] and Vénéreau [298].

4.2. **Abhyankar-Sathaye conjecture.** We suppose that k is algebraically closed and char $(k) = 0$. Abhyankar [2] asked several questions about the existence of non-equivalent subspaces in the affine n-space. The questions are generalized into the following one, which is called the *Abhyankar-Sathaye conjecture.*

Question. *Let X be a closed subvariety in $\mathbb{A}^n$ and let $\varphi : X \to \mathbb{A}^n$ be a closed embedding. Does there then exist an automorphism θ of $\mathbb{A}^n$ such that $\theta |_X = \varphi$?*

We say that X has *AM property* if there exists a required automorphism θ of $\mathbb{A}^n$. The answer to the question is negative in general (cf. Jelonek [118]), but there are several interesting subvarieties which have AM property.

(1) *Let X be a subvariety of dimension m in $\mathbb{A}^n$ such that X is isomorphic to $\mathbb{A}^m$ and $3m < n$. Then there exists an automorphism $f \in \mathrm{Aut}\,(\mathbb{A}^n)$ such that $f(X)$ is a linear subspace of $\mathbb{A}^n$. Furthermore, f is a composite of linear and de Jonquière transformations. As a consequence, each curve C in $\mathbb{A}^4$ isomorphic to $\mathbb{A}^1$ is rectifiable* (Craighero [41]; see Jelonek [118]).
(2) *Let $K_n = \{x_1x_2 \cdots x_n = 0\}$ be the union of n coordinate hyperplanes. Then K_n has AM property* (Jelonek [119]).
(3) *Let X be a subvariety in $\mathbb{A}^3$ isomorphic to $\mathbb{A}^2$. Suppose that X is invariant under a linear G_m-action on $\mathbb{A}^3$. Then there is a G_m-equivariant automorphism of $\mathbb{A}^3$ which carries X onto a coordinate hyperplane* (tom Dieck-Petrie [292]).
(4) *Let X be a subvariety in $\mathbb{A}^n$ isomorphic to $\mathbb{A}^{n-1}$. Suppose that X is invariant under an effective linear action of $T \times F$, where T is a torus of dimension $\geq n-2$ and F a finite abelian group. Then there is a $(T \times F)$-equivariant automorphism of $\mathbb{A}^n$ which carries X onto a coordinate hyperplane* (Russell [256]).
(5) *Let X be a hypersurface in $\mathbb{A}^3$ which is isomorphic to $\mathbb{A}^2$ and let V be the closure of X in $\mathbb{P}^3$ into which $\mathbb{A}^3$ is naturally embedded. Let H_∞ be the hyperplane at infinity and set $L = V \cap H_\infty$. If L is a line and V has multiplicity $\deg V - 1$ along L, then the Abhyankar-Sathaye conjecture holds true* (Kishimoto [150]).

(6) *Let the notations be the same as in* (5) *above. If* $\deg V \le 3$ *then the Abhyankar-Sathaye conjecture holds true* (Ohta[231]).

(7) *Let* X *and* Y *be closed algebraic subvarieties of the affine space* $\mathbb{A}^n$*, let* $\varphi : X \to Y$ *be an isomorphism and let* T_X *be the Zariski tangent bundle of* X*. If* $n > \max(2\dim X + 1, \dim T_X)$ *then* φ *can be extended to an automorphism of* $\mathbb{A}^n$ (Kaliman [121]).[22]

In the above question, suppose that X is isomorphic to $\mathbb{A}^{n-1}$ and φ is the embedding of $\mathbb{A}^{n-1}$ onto a coordinate hyperplane. If there exists an isomorphism $\theta \in \mathrm{Aut}\,(\mathbb{A}^n)$ with $\theta\,|_X = \varphi$, we say that X is *rectifiable*. We have several results about the rectifiability.

(8) *Let* X *be a hypersurface in* $\mathbb{A}^3$ *defined by* $f(x,y)z^n + g(x,y) = 0$*, where* $f, g \in k[x,y]$ *and* $n > 0$*. Suppose that* $X \cong \mathbb{A}^2$*.*[23] *Then* X *is rectifiable* (Sathaye [265], Wright [303]).

(9) *Let* X *be a hypersurface in* $\mathbb{A}^3$ *defined by* $f_0(x,y) + f_1(x,y)z + \cdots + f_n(x,y)z^n = 0$*, where* $f_0, \ldots, f_n \in k[x,y]$ *and* $\gcd(f_1, \ldots, f_n)$ *is a non-unit in* $k[x,y]$*. If* $X \cong \mathbb{A}^2$ *then* X *is rectifiable* (Russell-Sathaye [257])

(10) *Let* X *be a hypersurface in* $\mathbb{A}^4$ *defined by* $f(x,y)u + g(x,y,z) = 0$*. If* $X \cong \mathbb{A}^3$ *then every* X_α *defined by* $f(x,y)u + g(x,y,z) = \alpha$ $(\alpha \in k)$ *is isomorphic to* $\mathbb{A}^3$*.* But the rectifiability is not known in general (Kaliman-Vénéreau-Zaidenberg [129]).

(11) A Vénéreau polynomial v_n $(n \ge 1)$ is a polynomial in $\mathbb{C}[x,y,z,u]$ defined by $v_n := y + x^n(xz + y(yu + z^2))$. It is known by Vénéreau [298] that v_n is a coordinate of $\mathbb{C}[x,y,z,u]$ if $n \ge 3$, while the case $n = 1, 2$ is not known. See also Kaliman-Zaidenberg [133].

Related to the AMS theorem, there is the following problem.

Question. *Let* C *be a smooth curve on* $\mathbb{A}^2$ *which has one place at infinity. In the conjugate class of* C *by the automorphisms of* $\mathbb{A}^2$*, find a defining equation of* C *with minimal degree in* x *and* y*.*

Let g be the geometric genus of C. The case $g = 0$ is reduced to the AMS theorem. The standard form of C in the case $g = 1$ is given by $y^2 = x^3 + ax + b$. The case $g = 2, 3, 4, \ldots$ are treated by Neumann

[22]Gurjar informed me of an article by Srinivas [274] which contains Nori's proof of the fact that any two embeddings of a smooth affine variety into the affine space with codimension $> 2\dim X + 1$ are conjugate to each other under an automorphism of the affine space.

[23]It is shown in Kaliman-Zaidenberg [130] that $X \cong \mathbb{A}^2$ if and only if X is a homology plane. We also note that they define an affine modification to generalize the construction $\mathrm{Spec}\, k[x,y,g/f]$ and prove very nice algebro-geometric and topological properties of the affine modifications.

[227] by topological methods using splice diagram and in A'Campo-Oka [8] depending on Tschirnhausen resolution tower. In [194], the author treated the case $g = 2, 3, 4$ from the viewpoint of classifying the boundary graph of $\mathbb{A}^2$ when the base points of the linear pencil $\langle \overline{C}, d\ell_\infty \rangle$ is resolved, where $\overline{C}$ is the closure of C in $\mathbb{P}^2$, ℓ_∞ is the line at infinity and $d = \deg \overline{C}$. The existence theorem by Sathaye-Stenerson [265] of an affine plane curve with one place at infinity corresponding to a given characteristic δ-sequence plays a crucial role. This theorem is based on the theory of approximate roots by Abhyankar-Moh [5] and on the planer semigroup by Abhyankar-Singh [4]. Suzuki [283] provides another geometric interpretation of the theory of approximate roots. There are further results in Fujimoto-Suzuki [282], Nakazawa-Oka [226] and Oka [233].

Another variant of the AMS theorem is the following result of Lin-Zaidenberg [168].

Theorem 4.2.1. *Let C be an irreducible algebraic curve on the complex affine plane $\mathbb{A}^2$. Suppose that C is topologically contractible. Then C is defined by $x^m = y^n$ for $m, n > 0$ and $\gcd(m, n) = 1$ with respect to a suitable system of coordinates $\{x, y\}$.*

The original proof involves deep results from Teichmüller theory. An easy algebro-geometric proof is available in Gurjar-the author [91]. It might be too optimistic to ask if one can classify all contractible hypersurfaces in $\mathbb{A}^3$.

5. Etale endomorphisms of algebraic varieties and the Jacobian problem

In this section, the ground field k is an algebraically closed field of characteristic zero. We often confuse k with $\mathbb{C}$ if the use of $\mathbb{C}$ is more suitable.

5.1. **Jacobian conjecture.** There are two publications which provide an extensive coverage of the state of the conjecture. One is an article by Bass-Connell-Wright [25] and another is a monograph by Essen [59]. The Jacobian conjecture is stated as follows.

Jacobian Conjecture. *Let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ be polynomials such that the Jacobian determinant is everywhere nonzero on $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. Then the polynomial mapping $f = (f_1, \ldots, f_n) : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism.*

Non-vanishing of the Jacobian determinant implies via the inverse mapping theorem that f is locally isomorphic. Hence the core of the conjecture is that the *local* isomorphism at each point of $\mathbb{A}^n$ implies a *global* isomorphism. The conjecture is stated more geometrically as follows.

Second form. *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be an unramified endomorphism. Then f is an automorphism.*

Since a dominant unramified morphism of smooth algebraic varieties is a flat morphism, we say more often an *étale* endomorphism instead of an *unramified* endomorphism. If we look at the second form of the conjecture, we can think of the following generalization of the conjecture.

Generalized Jacobian Conjecture. *Let $X := \operatorname{Spec} A$ be a smooth affine variety of dimension n and let $f : X \to X$ be an étale endomorphism. Then f is a finite morphism.*[24]

It was first proved by Ax [22] that an injective endomorphism of an algebraic variety X is an automorphism. So, it suffices to show that $f : X \to X$ is injective if f is étale. Whenever we have to distinguish the source X from the target X, we write $f : X_1 \to X_2$, where $X_1 = X_2 = X$.

The following results are proved.

(1) *f is a quasi-finite morphism. If A is factorial, then the image $f(X)$ is an open set containing all codimension one points of X, i.e.,* $\operatorname{codim}_X(X \setminus f(X)) \geq 2$.
(2) *Either if f is birational or if $k(X_1)/k(X_2)$ is a Galois extension of the function fields and X is simply connected, then f is an automorphism* (cf. Abhyankar [1]).
(3) *Given a polynomial endomorphism $F : \mathbb{A}^n \to \mathbb{A}^n$, there exist an integer $m > 0$ and polynomial automorphisms $G, H : \mathbb{A}^{n+m} \to \mathbb{A}^{n+m}$ such that the composite $F' := G \circ F^{[m]} \circ H$ satisfies the condition* $\deg F' \leq 3$. *If $J(F; X) \in \mathbb{C}^*$ holds, so does the condition $J(F'; X^{[m]}) \in \mathbb{C}^*$. Hence the Jacobian conjecture holds in general if it holds for polynomial endomorphisms satisfying* $\deg F \leq 3$ (cf. Bass-Connell-Wright [25]).
(4) *With the notations in* (3), *if* $\deg F = 2$ *then the conjecture holds true* (Wang [300]).
(5) *The Jacobian conjecture holds in general if it holds for a polynomial endomorphism $F : \mathbb{A}^n \to \mathbb{A}^n$ of the special form $F =$*

[24] If we claim that f is an automorphism, there is an obvious counterexample. Namely, the m-th power endomorphism of G_m is étale but not an automorphism.

$X + K, K = (k_1, \ldots, k_n)$, where k_i is either 0 or a homogeneous polynomial of degree 3.

(6) *With the notations in (5), if the Jacobian conjecture holds for the k_i in the form $k_i = (\sum_{j=1}^{n} a_{ji}x_j)^3$, $a_{ji} \in \mathbb{C}$, then it holds in general. If the coefficient matrix $A = (a_{ji})$ of the linear form in K satisfies either $\operatorname{rank} A \leq 2$ or $n - \operatorname{rank} A \leq 2$, then F is an automorphism.* (Drużkowski [53, 54].)

When $X = \mathbb{A}^2$, we have more specific results. We write f, g and x, y instead of f_1, f_2 and x_1, x_2, respectively. We set $m = \deg f$ and $n = \deg g$. We assume that the Jacobian determinant $J((f, g)/(x, y)) \in k^*$.

(7) *If $\min(m, n) > 1$ then $\gcd(m, n) > 1$. Hence if either m or n is a prime number, then $k[x, y] = k[f, g]$* (Magnus [169]).

(8) *$k[x, y] = k[f, g]$ if (i) either $m = 4$ or $n = 4$, or (ii) $m = 2p > n$ and p is an odd prime* (Nakai-Baba [225]).

(9) *If either m or n is a product of at most two prime numbers, then $k[x, y] = k[f, g]$ holds* (Applegate-Onishi [12]).[25]

(10) *If $\max(m, n) \leq 100$, then $k[f, g] = k[x, y]$ holds* (Moh [214]).

(11) *If $\gcd(m, n) < 16$ then $k[x, y] = k[f, g]$* (Heitmann [111]).[26]

(12) *With the affine plane $\mathbb{A}^2$ embedded naturally into the projective plane $\mathbb{P}^2$ as the complement of the line at infinity ℓ_∞, denote by $\overline{V(f)}$ (resp. $\overline{V(g)}$) the closure in $\mathbb{P}^2$ of the affine plane curve $V(f)$ (resp. $V(g)$) defined by $f = 0$ (resp. $g = 0$). Suppose that $\overline{V(f)} \cap \ell_\infty$ as well as $\overline{V(g)} \cap \ell_\infty$ consists of a single point for every pair (f, g) satisfying $J(F; X) \in k^*$. Then the Jacobian conjecture in the case $n = 2$ holds* (Abhyankar [1]).

(13) *The Jacobian conjecture is equivalent to the following condition: For any pair of polynomials $F = (f, g)$ satisfying the condition $J(F; X) \in k^*$, the Newton polygon $N(f)$ (or $N(g)$) is a triangle with three points $(0, q), (0, 0), (p, 0)$ as summits, where p, q are non-negative integers.* (Abyankar [1]).[27]

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[25] Write $\deg f = dm'$ and $\deg g = dn'$ with $\gcd(m', n') = 1$. Magnus' theorem (see the above (7)) states that $k[x, y] = k[f, g]$ if $d = 1$. Applegate-Onishi's result was improved by A. Nowicki, Y. Nakai and M. Nagata to the theorem that *if $d \leq 8$ or d is a prime, then $k[x, y] = k[f, g]$* (cf. [229, 230, 223, 224]). In these papers, one can easily recognize the significance of the Newton polygons, edges and the gradings associated with the slopes of edges. A treatment similar to Applegate-Onishi's approach was taken by M. Oka [232]

[26] This result also gives a different approach to verify the above result of Moh.

[27] Given a polynomial $f = \sum_{i,j} c_{ij}x^i y^j$, the set $S(f) = \{(i, j)\ ;\ c_{ij} \neq 0\}$ is called the *support* of f. In the first quadrant of the coordinate plane, consider the smallest convex polygon containing the origin $(0, 0)$ and the points of $S(f)$. We call it the *Newton polygon* of f and denote it by $N(f)$. If the Jacobian determinant of f and

(14) *It suffices to prove the Jacobian conjecture for any pair of polynomials $F = (f, g)$ satisfying the condition $J(F; X) \in k^*$ and the additional assumption that the curves $f = \alpha$ are irreducible for all $\alpha \in k$* (Kaliman [123]).

(15) *Let $f : X \to X$ be an étale endomorphism. Suppose that f preserves a (possibly reducible) curve C, i.e., $f^{-1}(C) \subseteq C$. Then f is an automorphism* (the author-Masuda [199]). [28]

(16) *Let $f : X \to X$ be an étale endomorphism which commutes with an effective action of a finite group G on $\mathbb{A}^2$, i.e., ${}^g f(P) = f({}^g P)$ for $g \in G$ and $P \in \mathbb{A}^2$. Suppose that* $\dim(\mathbb{A}^2)^G \geq 1$, *where $(\mathbb{A}^2)^G$ is the fixed point locus of G. Then f is an automorphism* (the author-Masuda [199]).

(17) *There exists a pair of polynomials (f, g) in two variables x, y with real coefficients such that the polynomial mapping $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is a local isomorphism but not a homeomorphism. The Jacobian determinant is not invertible but does not have zeros in* $\mathbb{R}^2$ (Pinchuk [237]).

We shall note that there are completely new approaches by Kambayashi [137] (see also [141]). Namely, it is shown that *if the natural map from the ind-affine variety of automorphisms of $\mathbb{A}^n$ to the ind-affine variety of endomorphisms of $\mathbb{A}^n$ is a closed immersion, then every polynomial endomorphism of $\mathbb{A}^n$ with nowhere vanishing Jacobian determinant is an automorphism*. There are relationships (or equivalence) with the other conjectures. Essen [59] points out the equivalence between the Jacobian conjecture and the weak Dixmier conjecture concerning the surjectivity of a ring endomorphism of the Weyl algebra $\mathcal{D}_n$ preserving the Γ-filter. There is a link between the Jacobian conjecture and a conjecture in the representation theory of a compact connected Lie group (see Mathieu [178]).

5.2. **Generalized Jacobian Conjecture.** There is a definitive result by Iitaka [114]

Theorem 5.2.1. *Let X be a smooth algebraic variety of general type, i.e., $\overline{\kappa}(X) = \dim X$. Then any étale endomorphism $\varphi : X \to X$ is an automorphism.*

Hence we assume that $\overline{\kappa}(X) < \dim X$. Then we must confine ourselves to the case of dimension 2 to make use of the structure theorems.

g is a nonzero constant, then the Newton polygons $N(f)$ and $N(g)$ are similar to each other.

[28]The result relies on a paper by Aoki [10] which treats an irreducible affine plane curve $C \subset \mathbb{A}^2$ with $\overline{\kappa}(X) \leq 1$ and an étale endomorphism of X, where $X = \mathbb{A}^2 \setminus C$.

We begin with the affirmative results. For the references, see the author [197], Gurjar-the author [94] and the author-Masuda [199].

Theorem 5.2.2. *Let X be a smooth affine surface with $\overline{\kappa}(X) = -\infty$. Suppose either that* (1) *X is irrational but not elliptic ruled, or that* (2) $\Gamma(X, \mathcal{O}_X)^* \neq k^*$ *and* $\operatorname{rank}(\Gamma(X, \mathcal{O}_X)^*/k^*) \geq 2$ *if X is rational. Then any étale endomorphism $\varphi : X \to X$ is an automorphism.*

Theorem 5.2.3. *Let X be a smooth affine surface with an $\mathbb{A}^1_*$-fibration $\rho : X \to C$. Assume that $\operatorname{Pic} X = (0)$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$. Then any étale endomorphism $\varphi : X \to X$ is an automorphism provided one of the following conditions are satisfied.*

(1) *$\rho \circ \varphi = \rho$.*

(2) *$\overline{\kappa}(X) = 1$ and there are at least three singular fibers of ρ whose multiplicity sequence is none of the following:*

$$\{m_1, \dots, m_r\} = \{2, 2, 2, 2\}, \{2, 3, 6\}, \{2, 4, 4\}, \{3, 3, 3\}$$

(3) *X is a $\mathbb{Q}$-homology plane with $\overline{\kappa}(X) = 1$.*[29]

Theorem 5.2.4. *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \dots, m_nA_n$ exhaust all multiple fibers of ρ. Suppose that either $n \geq 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$. Then any étale endomorphism $\varphi : X \to X$ is an automorphism.*

The generalized Jacobian Conjecture has counterexamples. The first one of them is given in [197]. It is the complement $X := \mathbb{P}^2 - C$ of a smooth cubic curve C. This X has a *non-finite* étale endomorphism of degree 3. In fact, let $\widetilde{X}$ be the universal covering space of X. Then X is embedded into $\widetilde{X}$ as the complement of six irreducible curves which are isomorphic to the affine line. Furthermore, there is a counterexample which is related to Theorem 5.2.4. Namely, there is a $\mathbb{Q}$-homology plane X with an $\mathbb{A}^1$-fibration ρ which has two irreducible multiple fibers of multiplicities $\{2, 2\}$ and which has a non-finite étale endomorphism of degree 2. We remark the following result.

Theorem 5.2.5. *Let X be a smooth algebraic variety and let $\varphi : X \to X$ be a non-finite étale endomorphism. Suppose that $\pi_1(X)$ is a finite group. Let $q : \widetilde{X} \to X$ be the universal covering morphism. Then the following assertions hold:*

(1) *There exists a non-finite étale endomorphism $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ such that $q \circ \widetilde{\varphi} = \varphi \circ q$.*

(2) *There exists a group endomorphism $\chi : \pi_1(X) \to \pi_1(X)$ such that $\widetilde{\varphi}(gu) = \chi(g)\widetilde{\varphi}(u)$ for any $g \in \pi_1(X)$ and $u \in \widetilde{X}$.*

[29] We can drop the condition $\operatorname{Pic}(X) = (0)$ in this case, though $\operatorname{Pic}(X) \otimes \mathbb{Q} = (0)$.

(3) *Conversely, if there exists a non-finite étale endomorphism* $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ *satisfying the condition that* $\widetilde{\varphi}(gu) = \chi(g)\widetilde{\varphi}(u)$ *with a group endomorphism* $\chi : \pi_1(X) \to \pi_1(X)$ *for* $g \in \pi_1(X)$ *and* $u \in \widetilde{X}$*, then there exists a non-finite étale endomorphism* $\varphi : X \to X$ *such that* $q \circ \widetilde{\varphi} = \varphi \circ q$.

Suppose that $\varphi : X \to X$ is a non-finite étale endomorphism of a smooth affine surface X. Write it as $\varphi : X_1 \to X_2$ and let $\widetilde{X}_2$ be the normalization of X_2 in the function field $k(X_1)$. Then X_1 is an open set of $\widetilde{X}_2$ and φ is the restriction of the normalization morphism $\widetilde{\varphi} : \widetilde{X}_2 \to X_2$. Then we are interested in the complement $\widetilde{X}_2 \setminus X_1$. In most cases, it seems that $\widetilde{X}_2 \setminus X_1$ is a disjoint union of irreducible components which are isomorphic to the affine line (or contractible curves). In fact, we have the following result (cf. the author-Masuda [199]).

Theorem 5.2.6. *Let* X *be a smooth affine surface with* $\overline{\kappa}(X) = 1$ *and let* $\varphi : X \to X$ *be an étale endomorphism. Then the following assertions hold:*

(1) $\widetilde{X}_2$ *is a smooth affine surface with* $\overline{\kappa}(\widetilde{X}_2) = 1$*. Furthermore,* $\widetilde{\varphi} : \widetilde{X}_2 \to X_2$ *is an étale Galois covering with a cyclic group* G *of order* $n := \deg \varphi$ *as the Galois group.*

(2) *The* $\mathbb{A}^1_*$*-fibration* $\rho_1 : X_1 \to C_1$*, which exists uniquely upto automorphisms because of* $\overline{\kappa}(X) = 1$*, extends to an* $\mathbb{A}^1_*$*-fibration* $\widetilde{\rho}_1 : \widetilde{X}_2 \to C_1$ *such that* $\beta \circ \widetilde{\rho}_1 = \rho_2 \circ \widetilde{\varphi}$*, where* $\beta : C_1 \to C_2$[30] *is an automorphism.*

(3) $\widetilde{X}_2 - X_1$ *is a disjoint union of irreducible curves isomorphic to the affine line. The number* N *of irreducible components of* $\widetilde{X}_2 - X_1$ *is zero or given by the following formula:*

$$N = \sum_{i=1}^{r} (nd_i - d'_i),$$

where r *is the number of singular fibers of* ρ_2 *(and hence of* ρ_1*) and* d_i *(resp.* d'_i*) is the number of irreducible components in* $\rho_2^*(P_i)$ *(resp.* $\rho_1^*(\beta^{-1}(P_i))$*) isomorphic to* $\mathbb{A}^1$ *with* $\{P_1, \ldots, P_r\}$ *exhausting all points of* C_2 *such that* $\rho_2^*(P_i)$ *is a singular fiber.*

Comments. (1) The validity of the generalized Jacobian conjecture for affine algebraic surfaces X which are close to $\mathbb{A}^2$ will undoutedly confirm our belief in the Jacobian conjecture itself. For this purpose,

[30] We denote by $\rho_i : X_i \to C_i$ $(i = 1, 2)$ the unique $\mathbb{A}^1_*$-fibration on X.

the best candidates, for the moment, seem to be (1) the Platonic $\mathbb{A}^1$-fiber space that is given as $\mathbb{A}^2/G$ minus the unique singular point, where G is a small finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ and (2) the complement $\mathbb{P}^2 \setminus C$, where C is a rational cuspidal curve defined by $X_0 X_1^{d-1} = X_2^d$ with $d \geq 2$. An étale endomorphism φ of X in the case (1) lifts up to an étale endomorphism $\widetilde{\varphi}$ of the universal covering space of X (and hence on $\mathbb{A}^2$) which commutes with the action of G. Namely, we are reduced to consider the Jacobian conjecture for $\mathbb{A}^2$ together with a G-action. In the case (2), X has an $\mathbb{A}^1$-fibration $\rho : X \to \mathbb{A}^1$ whose fibers are all irreducible and reduced except for one multiple fiber of multiplicity d.

(2) Generalizing the case (2), one can define an affine pseudo-plane as a smooth affine surface X defined over $\mathbb{C}$ equipped with an $\mathbb{A}^1$-fibration $\rho : X \to B \cong \mathbb{A}^1$ such that all fibers of ρ are irreducible and reduced except for one irreducible multiple fiber. Such a surface appears when one considers a splitting of a given étale endomorphism $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ as $\varphi : \mathbb{A}^2 \xrightarrow{\varphi_1} X \xrightarrow{\varphi_2} \mathbb{A}^2$. We refer to the author [175, 197, 198] for the details.

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School of Science and Technology, Kwansei Gakuin University, 2-1 Gakuen, Sanda, Hyogo 669–1337, Japan

E-mail address: `miyanisi@ksc.kwansei.ac.jp`